Durham University
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Epiphany 2020

# Riemannian Geometry IV, Homework 2 (Week 12)

**Due date** for starred problems: **Friday, January 31**.

**2.1.** ($\star$) Let $H_3(\mathbb{R})$ be the set of $3\times 3$ unit upper-triangular matrices (i.e. the matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_1, x_2, x_3 \in \mathbb{R}$).

(a) Show that $H_3(\mathbb{R})$ is a group with respect to matrix multiplication. This group is called the *Heisenberg group.*

(b) Show that the Heisenberg group is a Lie group. What is its dimension?

(c) Prove that the matrices

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis of the tangent space $T_eH_3(\mathbb{R})$ of the group $H_3(\mathbb{R})$ at the neutral element $e$.

(d) For each $k = 1, 2, 3$, find an explicit formula for the curve $c_k : \mathbb{R} \to H_3(\mathbb{R})$ given by $c_k(t) = \mathrm{Exp}\,(tX_k)$.

**2.2.** (a) Let $A, B \in M_n(\mathbb{R})$, $[A, B] = 0$. Take $t \in \mathbb{R}$ and show that $\mathrm{Exp}\,(t(A + B)) = \mathrm{Exp}\,(tA)\,\mathrm{Exp}\,(tB)$ (in particular, you obtain that $\mathrm{Exp}\,(A + B) = \mathrm{Exp}\,(A)\,\mathrm{Exp}\,(B)$).

(b) Show that

$$\mathrm{Exp}\left( t \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Guess what would be the exponential of an $n \times n$-matrix of the same form (i.e., a Jordan block with zero eigenvalue).

(c) Show that

$$\mathrm{Exp}\left( t \begin{pmatrix} c & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & c \end{pmatrix} \right) = e^{tc} \begin{pmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**2.3.** The *special unitary group* $SU_n \subset M_n(\mathbb{C})$ consists of $n \times n$ matrices $A$ with complex entries and unit determinant satisfying the equation $\bar{A}^t A = I = A\bar{A}^t$.

(a) Show that $SU_n$ forms a group under matrix multiplication.

(b) Show that $SU_2$ consists of all matrices of the form

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}, \quad z, w \in \mathbb{C}, \quad |z|^2 + |w|^2 = 1.$$

(c) Show that $SU_2$ is a smooth (real) manifold. Find its dimension.

(d) Show that $SU_2$ is a Lie group.

(e) Find the Lie algebra $\mathfrak{su}_2$ of $SU_2$ as a subspace of $M_2(\mathbb{C})$. Find any basis $\{v_1, v_2, v_3\}$ of $\mathfrak{su}_2$. Compute explicitly the left-invariant vector fields $X_1, X_2, X_3$ on $SU_2$ such that $X_i(I) = v_i$.