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## Riemannian Geometry IV, Solutions 3 (Week 13)

**3.1.** ($\star$) Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with a *bi-invariant* Riemannian metric (i.e., both $L_g$ and $R_g$ are isometries for every $g \in G$). Let $\mathfrak{g}$ denote the Lie algebra of $G$, and let $X, Y, Z \in \mathfrak{g}$.

(a) Show that $\langle X, Y \rangle$ is a constant function on $G$.

(b) Use the relation

$$\langle Z, \nabla_X Y \rangle = \frac{1}{2} \left( X\langle Z, Y \rangle + Y\langle Z, X \rangle - Z\langle Y, X \rangle + \langle X, [Z, Y] \rangle + \langle Y, [Z, X] \rangle - \langle Z, [Y, X] \rangle \right)$$

and the fact that the metric is left-invariant to prove that $\langle Z, \nabla_Y Y \rangle = \langle Y, [Z, Y] \rangle$.

(c) By Corollary 6.18, the bi-invariance of the metric implies that

$$\langle [U, X], V \rangle = -\langle U, [V, X] \rangle$$

for $X, U, V \in \mathfrak{g}$. Use this fact to conclude that $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$.

(d) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

*Solution:*

(a)

$$\langle X(g), Y(g) \rangle_g = \langle DL_g(e)X(e), DL_g(e)Y(e) \rangle_g = \langle X(e), Y(e) \rangle_e,$$

so $\langle X(g), Y(g) \rangle_g$ does not depend on $g$.

(b) The relation with 6 terms in the RHS implies that

$$\langle Z, \nabla_Y Y \rangle = \frac{1}{2} \left( Y\langle Z, Y \rangle + Y\langle Z, Y \rangle - Z\langle Y, Y \rangle + \langle Y, [Z, Y] \rangle + \langle Y, [Z, Y] \rangle - \langle Z, [Y, Y] \rangle \right) =$$
$$\frac{1}{2} \left( \langle Y, [Z, Y] \rangle + \langle Y, [Z, Y] \rangle \right),$$

since the first three derivatives of the right hand side of the relation vanish by (a). Moreover, we have $[Y, Y] = 0$. Thus, we conclude that

$$\langle Z, \nabla_Y Y \rangle = \langle Y, [Z, Y] \rangle.$$

(c) The bi-invariance implies that

$$\langle [Y, X], Y \rangle = -\langle Y, [Y, X] \rangle = -\langle [Y, X], Y \rangle,$$

so $\langle [Y, X], Y \rangle = 0$. This gives us $\langle X, \nabla_Y Y \rangle = 0$ for all left-invariant $X$, so we have $\nabla_Y Y = 0$ for all left-invariant $Y$.

(d) We calculate

$$0 = \nabla_{X+Y}(X + Y) = \nabla_X Y + \nabla_Y X + \nabla_X X + \nabla_Y Y = \nabla_X Y + \nabla_Y X = 2\nabla_X Y - [X, Y].$$

Division by two finally yields

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

**3.2.** Let $(M, g)$ be a Riemannian manifold and $R$ its curvature tensor. Let $f, g, h \in C^\infty(M)$, and $X, Y, Z, W$ be vector fields on $M$. Show that

(a) $R(fX, Y)Z = fR(X, Y)Z$;

(b) $R(X, fY)Z = fR(X, Y)Z$;

(c) $\langle R(X,Y)fZ, W\rangle = \langle fR(X,Y)Z, W\rangle$;

(d) $R(fX, gY)hZ = fghR(X,Y)Z$.

*Solution:*

(a) Note that $[fX, Y] = f[X,Y] - (Yf)X$. We have

$$\begin{aligned} R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]}Z = \\ &= f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-(Yf)X}Z = \\ &= f\nabla_X\nabla_Y Z - (Yf)\nabla_X Z - f\nabla_Y\nabla_X Z - f\nabla_{[X,Y]}Z + (Yf)\nabla_X Z = \\ &= f(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) = fR(X,Y)Z. \end{aligned}$$

(b) Using the symmetry $R(X,Y)Z = -R(Y,X)Z$ and applying (a) we obtain

$$R(X, fY)Z = -R(fY, X)Z = -fR(Y,X)Z = fR(X,Y)Z.$$

(c) Using the symmetry $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$ twice, we obtain

$$\begin{aligned} \langle R(X,Y)fZ, W\rangle &= \langle R(fZ,W)X, Y\rangle = \langle fR(Z,W)X, Y\rangle = \\ &= f\langle R(Z,W)X, Y\rangle = f\langle R(X,Y)Z, W\rangle = \langle fR(X,Y)Z, W\rangle. \end{aligned}$$

(d) Since (c) holds for all vector fields $W$, we conclude that

$$R(X,Y)fZ = fR(X,Y)Z.$$

Using this together with (a) and (b), we obtain

$$R(fX, gY)hZ = fghR(X,Y)Z.$$

**3.3.** ($\star$) **First Bianchi Identity**
Let $(M, g)$ be a Riemannian manifold and $R$ its curvature tensor. Prove the *First Bianchi Identity*:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

for $X, Y, Z$ vector fields on $M$ by reducing the equation to *Jacobi identity*

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

*Solution:* We have

$$\begin{aligned} &R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = \\ &= (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z) + (\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]}X) + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]}Y) = \\ &= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X) - (\nabla_{[X,Y]}Z) + \nabla_{[Y,Z]}X) + \nabla_{[Z,X]}Y) = \\ &= \nabla_X[Y,Z] + \nabla_Y[Z,X] + \nabla_Z[X,Y] - (\nabla_{[X,Y]}Z) + \nabla_{[Y,Z]}X) + \nabla_{[Z,X]}Y) = \\ &= (\nabla_X[Y,Z] - \nabla_{[Y,Z]}X) + (\nabla_Y[Z,X] - \nabla_{[Z,X]}Y) + (\nabla_Z[X,Y] - \nabla_{[X,Y]}Z) = \\ &= -([[Y,Z],X] + [[Z,X],Y] + [[X,Y],Z]) = 0. \end{aligned}$$

**3.4.** ($\star$) Parametrize the sphere $S^2_r$ of radius $r$ in $\mathbb{R}^3$ by

$$(x,y,z) = (r\cos\varphi\sin\vartheta,\ r\sin\varphi\sin\vartheta,\ r\cos\vartheta),$$

and consider the metric on $S^2_r$ induced by the Euclidean metric in $\mathbb{R}^3$.

(a) Compute $R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi})$.

(b) Compute the *sectional curvature of* $S_r^2$, $K = \frac{R(\frac{\partial}{\partial\varphi},\frac{\partial}{\partial\vartheta},\frac{\partial}{\partial\vartheta},\frac{\partial}{\partial\varphi})}{\langle\frac{\partial}{\partial\varphi},\frac{\partial}{\partial\varphi}\rangle\langle\frac{\partial}{\partial\vartheta},\frac{\partial}{\partial\vartheta}\rangle-\langle\frac{\partial}{\partial\varphi},\frac{\partial}{\partial\vartheta}\rangle^2}$.

*Solution:*

(a) First, we compute Christoffel symbols.
Since

$$\frac{\partial}{\partial\varphi} = (-r\sin\varphi\sin\vartheta, r\cos\varphi\sin\vartheta, 0),$$

$$\frac{\partial}{\partial\vartheta} = (r\cos\varphi\cos\vartheta, r\sin\varphi\cos\vartheta, -r\sin\vartheta),$$

we have $(g_{ij}) = \begin{pmatrix} r^2\sin^2\vartheta & 0 \\ 0 & r^2 \end{pmatrix}$ and $(g^{ij}) = \begin{pmatrix} \frac{1}{r^2\sin^2\vartheta} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$.
So, $g_{11,2} = 2r^2\sin\vartheta\cos\vartheta$ and $g_{ij,k} = 0$ for all other choices of $i, j, k$. Since the metric is diagonal, we have $\Gamma_{ij}^k = \frac{1}{2}g^{kk}(g_{ik,j} + g_{kj,i} - g_{ij,k})$ which is non-zero only if the (unordered) triple $(i, j, k)$ coincides with $(1, 1, 2)$. This implies

$$\Gamma_{11}^2 = \frac{1}{2}\frac{1}{r^2}(-2r^2\sin\vartheta\cos\vartheta) = -\sin\vartheta\cos\vartheta,$$

$$\Gamma_{12}^1 = \frac{1}{2}\frac{1}{r^2\sin^2\vartheta}(2r^2\sin\vartheta\cos\vartheta) = \cot\vartheta.$$

By definition of Christoffel symbols,

$$\nabla_{\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\varphi} = \nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\vartheta} = \cot\vartheta\frac{\partial}{\partial\varphi},$$

$$\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} = -sin\vartheta\cos\vartheta\frac{\partial}{\partial\vartheta} \qquad \text{and} \qquad \nabla_{\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\vartheta} = 0.$$

$\nabla_{[\frac{\partial}{\partial\varphi},\frac{\partial}{\partial\vartheta}]}X = 0$ for any $X$ since $[\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}] = 0$.
Now we compute the Riemann curvature tensor.

$$\begin{aligned}
R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}) &= \langle R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\rangle = \\
&= \langle\nabla_{\frac{\partial}{\partial\varphi}}\nabla_{\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\varphi} - \nabla_{\frac{\partial}{\partial\vartheta}}\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} - \nabla_{[\frac{\partial}{\partial\varphi},\frac{\partial}{\partial\vartheta}]}\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\rangle = \\
&= \langle\nabla_{\frac{\partial}{\partial\varphi}}(\cot\vartheta)\frac{\partial}{\partial\varphi} - \nabla_{\frac{\partial}{\partial\vartheta}}(-\sin\vartheta\cos\vartheta\frac{\partial}{\partial\vartheta}), \frac{\partial}{\partial\vartheta}\rangle = \\
&= \langle(\frac{\partial}{\partial\varphi}\cot\vartheta)\frac{\partial}{\partial\vartheta} + \cot\vartheta\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} + (\frac{\partial}{\partial\vartheta}\sin\vartheta\cos\vartheta)\frac{\partial}{\partial\vartheta} + \sin\vartheta\cos\vartheta\nabla_{\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\vartheta}\rangle = \\
&= \langle\cot\vartheta(-\sin\vartheta\cos\vartheta)\frac{\partial}{\partial\vartheta} + (\cos^2\vartheta - \sin^2\vartheta)\frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\vartheta}\rangle = \\
&= -r^2\cos^2\vartheta + r^2(\cos^2\vartheta - \sin^2\vartheta) = -r^2\sin^2\vartheta.
\end{aligned}$$

(b)

$$K = \frac{\langle R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})\frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi}\rangle}{\|\frac{\partial}{\partial\varphi}\|^2\|\frac{\partial}{\partial\vartheta}\|^2 - \langle\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\rangle} = \frac{r^2\sin^2\vartheta}{r^2\sin^2\vartheta\cdot r^2} = \frac{1}{r^2}.$$