Durham University
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Epiphany 2020

Riemannian Geometry IV, Homework 3 (Week 13)

Due date for starred problems: **Friday, February 14**.

3.1. ($\star$) Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with a *bi-invariant* Riemannian metric (i.e., both L_g and R_g are isometries for every $g \in G$). Let $\mathfrak{g}$ denote the Lie algebra of G, and let $X, Y, Z \in \mathfrak{g}$.

(a) Show that $\langle X, Y \rangle$ is a constant function on G.

(b) Use the relation

$$\langle Z, \nabla_X Y \rangle = \frac{1}{2}\left(X\langle Z, Y \rangle + Y\langle Z, X \rangle - Z\langle Y, X \rangle + \langle X, [Z, Y] \rangle + \langle Y, [Z, X] \rangle - \langle Z, [Y, X] \rangle \right)$$

and the fact that the metric is left-invariant to prove that $\langle Z, \nabla_Y Y \rangle = \langle Y, [Z, Y] \rangle$.

(c) By Corollary 6.18, the bi-invariance of the metric implies that

$$\langle [U, X], V \rangle = -\langle U, [V, X] \rangle$$

for $X, U, V \in \mathfrak{g}$. Use this fact to conclude that $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$.

(d) Show that $\nabla_X Y = \frac{1}{2}[X, Y]$.

3.2. Let (M, g) be a Riemannian manifold and R its curvature tensor. Let $f, g, h \in C^\infty(M)$, and X, Y, Z, W be vector fields on M. Show that

(a) $R(fX, Y)Z = fR(X, Y)Z$;

(b) $R(X, fY)Z = fR(X, Y)Z$;

(c) $\langle R(X, Y)fZ, W \rangle = \langle fR(X, Y)Z, W \rangle$;

(d) $R(fX, gY)hZ = fghR(X, Y)Z$.

3.3. First Bianchi Identity

Let (M, g) be a Riemannian manifold and R its curvature tensor. Prove the *First Bianchi Identity*:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

for X, Y, Z vector fields on M by reducing the equation to *Jacobi identity*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

3.4. ($\star$) Parametrize the sphere S_r^2 of radius r in $\mathbb{R}^3$ by

$$(x, y, z) = (r\cos\varphi\sin\vartheta,\ r\sin\varphi\sin\vartheta,\ r\cos\vartheta),$$

and consider the metric on S_r^2 induced by the Euclidean metric in $\mathbb{R}^3$.

(a) Compute $R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi})$.

(b) Compute the *sectional curvature of* S_r^2, $K = \frac{R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\varphi})}{\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi} \rangle \langle \frac{\partial}{\partial\vartheta}, \frac{\partial}{\partial\vartheta} \rangle - \langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta} \rangle^2}$.