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Epiphany 2020

Riemannian Geometry IV, Solutions 4 (Week 14)

4.1. Scalar curvature
The *scalar curvature* $s(p)$ at point $p \in M$ is defined by

$$s(p) = \sum_{j=1}^{n} Ric_p(u_j),$$

where $\{u_j\}$ is an orthonormal basis of $T_p(M)$.

(a) Let V be a vector space, $\langle \cdot , \cdot \rangle$ is an inner product on V, and Q is a quadratic form on V. Show that there exists a linear map $T \in \mathrm{End}(V)$ such that $Q(x) = \langle Tx , x \rangle$ for every $x \in V$.

(b) Show that the scalar curvature is well-defined, i.e. it does not depend on the choice of an orthonormal basis of $T_p(M)$.

Solution:

(a) Choose any orthonormal basis $\{e_i\}$ of V. Then $Q(x)$ can be written as $Q(x) = x^t G x$ for appropriate symmetric matrix $G = (g_{ij})$. Here $g_{ij} = \tilde{Q}(e_i, e_j)$, where $\tilde{Q}$ is the symmetric bilinear form constructed by Q, i.e. $\tilde{Q}(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y))$.
Since the basis is orthonormal, the inner product can be written as $\langle x , x \rangle = x^t x$. We need to find (a matrix) T such that $Q(x) = \langle Tx , x \rangle$, i.e. $x^t G x = (Tx)^t x$, or equivalently, $x^t G x = x^t T^t x$. This holds if $G = T^t$, or $T = G^t (= G$ since G is symmetric). It is easy to check that T is well-defined: if we change basis via an orthogonal transformation matrix P, then G in the new basis becomes PGP^t, and T becomes PTP^{-1}, which agree since $P^t = P^{-1}$ for orthogonal matrices.

(b) According to Lemma 7.9 from the lectures, Ric_p is a quadratic form. Thus, (a) implies that there exists $T \in \mathrm{End}(T_pM)$ such that $Ric_p(u) = \langle Tu , u \rangle$ for every $u \in T_pM$. Denote the matrix of T in the basis $\{u_j\}$ by (T_{ij}). Then

$$s(p) = \sum_{j=1}^{n} Ric_p(u_j) = \sum_{j=1}^{n} \langle Tu_j , u_j \rangle = \sum_{j=1}^{n} \langle \sum_{i=1}^{n} T_{ij} u_i , u_j \rangle = \sum_{j=1}^{n} T_{jj} = \mathrm{tr}\,(T)$$

which does not depend on the basis.

4.2. A Riemannian manifold (M, g) is called an *Einstein manifold* if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v, w) = c\langle v, w\rangle$$

for every $p \in M$, $v, w \in T_pM$.

(a) Show that (M, g) is an Einstein manifold if and only if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v) = c$$

for every $p \in M$ and unit tangent vector $v \in T_pM$.

(b) Show that if (M, g) is of constant sectional curvature then (M, g) is Einstein manifold.

Solution: We have seen in class that $Ric_p(v, w)$ is a symmetric bilinear form on T_pM, and thus $Ric_p(v)$ is a quadratic form.

(a) If M is Einstein manifold, then

$$Ric_p(v) = Ric_p(v, v) = c\langle v, v\rangle,$$

which is equal to c for any unit vector v.
Conversely, if $Ric_p(v) = c$ for any unit vector v, then, by linearity,

$$Ric_p(\lambda v) = c\lambda^2 = c\langle \lambda v, \lambda v\rangle,$$

which implies

$$Ric_p(u) = c\langle u, u\rangle$$

for arbitrary vector $u \in T_pM$. Now, reconstructing symmetric bilinear form $Ric_p(v, w)$ by quadratic form $Ric_p(v) = Ric_p(v, v)$, we obtain

$$Ric_p(v, w) = \frac{1}{2}(Ric_p(v + w, v + w) - Ric_p(v) - Ric_p(w)) =$$
$$= \frac{1}{2}(c\langle v + w, v + w\rangle - c\langle v, v\rangle - c\langle w, w\rangle) = c\langle v, w\rangle$$

(b) Let M be n-dimensional, $p \in M$, and assume $K(\Pi) = K_0$ for all 2-dimensional subspaces Π of TM. Take arbitrary unit vector $v \in T_pM$, extend it to an orthonormal basis $\{v, v_2, \dots, v_n\}$. Then

$$Ric_p(v) = \sum_{i=2}^{n} K(v, v_i) = (n - 1)K_0,$$

so M is Einstein manifold.

4.3. Let (M, g) be a Riemannian manifold. The goal of this exercise is to show that M is of constant sectional curvature K_0 if and only if

$$\langle R(v_1, v_2)v_3, v_4\rangle = -K_0(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle)$$

for any $p \in M$ and $v_1, v_2, v_3, v_4 \in T_pM$. Denote the expression $-K_0\left(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_2, v_3\rangle\langle v_1, v_4\rangle\right)$ by (v_1, v_2, v_3, v_4).

(a) Show that if

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in T_pM$, then M is of constant sectional curvature K_0.

Now assume that M is of constant sectional curvature K_0. Our aim is to show that

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in T_pM$.

(b) Show that the expression (v_1, v_2, v_3, v_4) is a tensor, i.e. it is multilinear.

(c) Show that (v_1, v_2, v_3, v_4) has the same symmetries as Riemann curvature tensor has. Namely,
- $(v_1, v_2, v_3, v_4) = -(v_2, v_1, v_3, v_4)$
- $(v_1, v_2, v_3, v_4) = -(v_1, v_2, v_4, v_3)$
- $(v_1, v_2, v_3, v_4) = (v_3, v_4, v_1, v_2)$
- $(v_1, v_2, v_3, v_4) + (v_2, v_3, v_1, v_4) + (v_3, v_1, v_2, v_4) = 0$

(d) Show that if $\{v_1, v_2, v_3, v_4\} \subset \{v, w\}$, i.e. no more than two distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(e) Show that if no more than three distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(f) Show that for any four vectors $\{v_1, v_2, v_3, v_4\}$

$$\langle R(v_1, v_2)v_3, v_4\rangle - (v_1, v_2, v_3, v_4) = \langle R(v_3, v_1)v_2, v_4\rangle - (v_3, v_1, v_2, v_4),$$

i.e. the difference above is invariant with respect to cyclic permutation of first three arguments.

(g) Use Bianchi identity to prove the initial statement.

Solution:

(a) If the equality holds, we have

$$K(v, u) = \frac{\langle R(v, u)u, v\rangle}{\|v\|^2\|u\|^2 - \langle v, u\rangle^2} = \frac{K_0(\langle v, v\rangle\langle u, u\rangle - \langle v, u\rangle\langle u, v\rangle)}{\|v\|^2\|u\|^2 - \langle v, u\rangle^2} = K_0$$

(b) This can be seen from the explicit formula for (v_1, v_2, v_3, v_4).

(c) Straighforward calculations using the definition of (v_1, v_2, v_3, v_4).

(d) By the definition of sectional curvature,

$$\langle R(v, u)u, v\rangle = K_0 \left(\|v\|^2\|u\|^2 - \langle v, u\rangle^2\right) = (v, u, u, v).$$

For collections of vectors ordered in other way the statement follows by (c).

(e) Using linearity and (d), we obtain

$$\begin{aligned}\langle R(v_1, v_2 + v_3)v_2 + v_3, v_1\rangle &= (v_1, v_2 + v_3, v_2 + v_3, v_1) = \\ &= (v_1, v_2, v_2, v_1) + (v_1, v_2, v_3, v_1) + (v_1, v_3, v_2, v_1) + (v_1, v_3, v_3, v_1),\end{aligned}$$

and on the other side

$$\begin{aligned}&\langle R(v_1, v_2 + v_3)v_2 + v_3, v_1\rangle = \\ &= \langle R(v_1, v_2)v_2, v_1\rangle + \langle R(v_1, v_2)v_3, v_1\rangle + \langle R(v_1, v_3)v_2, v_1\rangle + \langle R(v_1, v_3)v_3, v_1\rangle = \\ &= (v_1, v_2, v_2, v_1) + \langle R(v_1, v_2)v_3, v_1\rangle + \langle R(v_1, v_3)v_2, v_1\rangle + (v_1, v_3, v_3, v_1),\end{aligned}$$

which leads to

$$\langle R(v_1, v_2)v_3, v_1\rangle + \langle R(v_1, v_3)v_2, v_1\rangle = (v_1, v_2, v_3, v_1) + (v_1, v_3, v_2, v_1). \tag{1}$$

By the symmetries, we obtain

$$\langle R(v_1, v_2)v_3, v_1\rangle = \langle R(v_1, v_3)v_2, v_1\rangle\,,$$

and the same holds for $(\cdot, \cdot, \cdot, \cdot)$, so (1) simplifies to

$$2\,\langle R(v_1, v_2)v_3, v_1\rangle = 2(v_1, v_2, v_3, v_1).$$

(f) Using (e), we obtain on one side

$$\begin{aligned}\langle R(v_1 + v_4, v_2)v_3, v_1 + v_4\rangle &= (v_1 + v_4, v_2, v_3, v_1 + v_4) = \\ &= (v_1, v_2, v_3, v_1) + (v_1, v_2, v_3, v_4) + (v_4, v_2, v_3, v_1) + (v_4, v_2, v_3, v_4),\end{aligned}$$

and on the other side

$$\begin{aligned}&\langle R(v_1 + v_4, v_2)v_3, v_1 + v_4\rangle = \\ &= \langle R(v_1, v_2)v_3, v_1\rangle + \langle R(v_1, v_2)v_3, v_4\rangle + \langle R(v_4, v_2)v_3, v_1\rangle + \langle R(v_4, v_2)v_3, v_4\rangle = \\ &= (v_1, v_2, v_3, v_1) + \langle R(v_1, v_2)v_3, v_4\rangle + \langle R(v_4, v_2)v_3, v_1\rangle + (v_4, v_2, v_3, v_4).\end{aligned}$$

Comparing both expressions, we conclude that

$$\langle R(v_1, v_2)v_3, v_4\rangle + \langle R(v_4, v_2)v_3, v_1\rangle = (v_1, v_2, v_3, v_4) + (v_4, v_2, v_3, v_1).$$

This implies

$$\langle R(v_1, v_2)v_3, v_4\rangle - (v_1, v_2, v_3, v_4) = -\langle R(v_4, v_2)v_3, v_1\rangle + (v_4, v_2, v_3, v_1).$$

Using the symmetries, we derive

$$-\langle R(v_4, v_2)v_3, v_1\rangle = -\langle R(v_3, v_1)v_4, v_2\rangle = \langle R(v_3, v_1)v_2, v_4\rangle\,,$$

and the same identity for $(\cdot, \cdot, \cdot, \cdot)$, so we end up with

$$\langle R(v_1, v_2)v_3, v_4\rangle - (v_1, v_2, v_3, v_4) = \langle R(v_3, v_1)v_2, v_4\rangle - (v_3, v_1, v_2, v_4)$$

(g) Using (f) and Bianchi identity, we conclude that

$$\begin{aligned}3\left(\langle R(v_1,v_2)v_3,v_4\rangle-(v_1,v_2,v_3,v_4)\right)&=\left(\langle R(v_1,v_2)v_3,v_4\rangle-(v_1,v_2,v_3,v_4)\right)+\\&+\left(\langle R(v_3,v_1)v_2,v_4\rangle-(v_3,v_1,v_2,v_4)\right)+\left(\langle R(v_3,v_3)v_1,v_4\rangle-(v_2,v_3,v_1,v_4)\right)=\\&=\left(\langle R(v_1,v_2)v_3,v_4\rangle+\langle R(v_3,v_1)v_2,v_4\rangle+\langle R(v_2,v_3)v_1,v_4\rangle\right)-\\&-\left((v_1,v_2,v_3,v_4)+(v_3,v_1,v_2,v_4)+(v_2,v_3,v_1,v_4)\right)=0-0=0,\end{aligned}$$

which completes the proof.

4.4. Constant sectional curvature of hyperbolic 3-space

Let $\mathbb{H}^3=\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid x_3>0\}$ be the upper half-space model of the 3-dimensional hyperbolic space, i.e. its metric is defined by $g_{ij}=0$ for $i\neq j$, $g_{ii}=1/x_3^2$.

(a) Show that sectional curvatures $K(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})$, $K(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3})$ and $K(\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3})$ in $\mathbb{H}^3$ are equal to -1.

(b) Use (a) and the linearity of the Riemann curvature tensor to show that for any $p\in\mathbb{H}^3$ and $v_1,v_2,v_3,v_4\in T_p\mathbb{H}^3$

$$\langle R(v_1,v_2)v_3,v_4\rangle=-(\langle v_1,v_3\rangle\langle v_2,v_4\rangle-\langle v_1,v_4\rangle\langle v_2,v_3\rangle)$$

holds.

(c) Use (b) to show that 3-dimensional hyperbolic space $\mathbb{H}^3$ has constant sectional curvature -1.

(d) Show that n-dimensional hyperbolic space $\mathbb{H}^n=\{x\in\mathbb{R}^n\mid x_n>0\}$ with metric $g_{ij}=0$ for $i\neq j$, $g_{ii}=1/x_n^2$ has constant sectional curvature -1.

Solution:

(a) We can compute the Christoffel symbols in a standard way obtaining

$$\Gamma^3_{11}=\Gamma^3_{22}=\frac{1}{x_3},\quad \Gamma^3_{33}=\Gamma^1_{13}=\Gamma^1_{31}=\Gamma^2_{23}=\Gamma^2_{32}=-\frac{1}{x_3},$$

the remaining ones are zero. Using this, we can also compute that

$$\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1}=\nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2}=\frac{1}{x_3}\frac{\partial}{\partial x_3},\quad \nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_3}=-\frac{1}{x_3}\frac{\partial}{\partial x_3},\quad \nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2}=\nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_1}=0,$$
$$\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_3}=\nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_1}=-\frac{1}{x_3}\frac{\partial}{\partial x_1},\quad \nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_3}=\nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_2}=-\frac{1}{x_3}\frac{\partial}{\partial x_2}.$$

Now, we compute $K(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})$ and $K(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3})$.

$$\begin{aligned}K(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})&=\frac{\left\langle R(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1}\right\rangle}{\|\frac{\partial}{\partial x_1}\|^2\|\frac{\partial}{\partial x_2}\|^2-\left\langle\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right\rangle^2}=\\&=\frac{1}{\|\frac{\partial}{\partial x_1}\|^2\|\frac{\partial}{\partial x_3}\|^2}\left\langle\nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2}-\nabla_{\frac{\partial}{\partial x_2}}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2}-\nabla_{\left[\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2}\right]}\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1}\right\rangle=\\&=x_3^2x_3^2\left\langle\nabla_{\frac{\partial}{\partial x_1}}\frac{1}{x_3}\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_1}\right\rangle=x_3^4\left\langle\frac{1}{x_3}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_1}\right\rangle=\\&=x_3^4\left\langle-\frac{1}{x_3^2}\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_1}\right\rangle=-x_3^4\frac{1}{x_3^2}\frac{1}{x_3^2}=-1\end{aligned}$$

and

$$K(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3}) = \frac{\left\langle R(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3})\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_1}\right\rangle}{\|\frac{\partial}{\partial x_1}\|^2\|\frac{\partial}{\partial x_3}\|^2 - \left\langle \frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3}\right\rangle^2} =$$

$$= \frac{1}{\|\frac{\partial}{\partial x_1}\|^2\|\frac{\partial}{\partial x_3}\|^2}\left\langle \nabla_{\frac{\partial}{\partial x_1}}\nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_3} - \nabla_{\frac{\partial}{\partial x_3}}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_3} - \nabla_{\left[\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3}\right]}\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_1}\right\rangle =$$

$$= x_3^2x_3^2\left\langle -\nabla_{\frac{\partial}{\partial x_1}}\frac{1}{x_3}\frac{\partial}{\partial x_3} + \nabla_{\frac{\partial}{\partial x_3}}\frac{1}{x_3}\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_1}\right\rangle = x_3^4\left\langle -\frac{1}{x_3}\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_3}\frac{1}{x_3}\frac{\partial}{\partial x_1} + \frac{1}{x_3}\nabla_{\frac{\partial}{\partial x_3}}\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_1}\right\rangle =$$

$$= x_3^4\left\langle \frac{1}{x_3^2}\frac{\partial}{\partial x_1} - \frac{1}{x_3^2}\frac{\partial}{\partial x_1} + -\frac{1}{x_3^2}\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_1}\right\rangle = -x_3^4\frac{1}{x_3^2}\frac{1}{x_3^2} = -1$$

Computation of $K(\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3})$ is similar.

Remark. In fact, the plane spanned by vectors $\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3}$ is tangent to vertical hyperbolic plane $x_2 = c$, so the corresponding sectional curvature is exactly the curvature of hyperbolic plane which is equal to -1. Thus, we could avoid the computation of $K(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_3})$.

(b) By computations similar to ones done in (a), we obtain that

$$\left\langle R(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_1}\right\rangle = \left\langle R(\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_1})\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_2}\right\rangle = \left\langle R(\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_1})\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right\rangle = 0$$

Now we see that for all vectors $\{v_1,v_2,v_3,v_4\}\subset\{\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\}$ we have an equality

$$\langle R(v_1,v_2)v_3,v_4\rangle = -\left(\langle v_1,v_4\rangle\langle v_2,v_3\rangle - \langle v_1,v_3\rangle\langle v_2,v_4\rangle\right)$$

By linearity, the equality above holds for any quadruple of tangent vectors.

(c) This follows from (b) and Exercise 4.3.

(d) It is easy to see that the Christoffel symbol Γ_{ij}^k is not zero if and only if one of (i,j,k) equals n and two others are equal. This implies that if all of (i,j,k,l) are distinct then R_{ijkl} vanishes. Applying the arguments of (b) we conclude that $\mathbb{H}^n$ has constant sectional curvature -1.