Durham University
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Epiphany 2020

Riemannian Geometry IV, Homework 4 (Week 14)

Due date for starred problems: **Friday, February 14**.

4.1. Scalar curvature

The *scalar curvature* $s(p)$ at point $p \in M$ is defined by

$$s(p) = \sum_{j=1}^{n} Ric_p(u_j),$$

where $\{u_j\}$ is an orthonormal basis of $T_p(M)$.

(a) Let V be a vector space, $\langle \cdot\,, \cdot \rangle$ is an inner product on V, and Q is a quadratic form on V. Show that there exists a linear map $T \in \mathrm{End}(V)$ such that $Q(x) = \langle Tx\,, x\rangle$ for every $x \in V$.

(b) Show that the scalar curvature is well-defined, i.e. it does not depend on the choice of an orthonormal basis of $T_p(M)$.

4.2. ($\star$) **Einstein manifolds**

A Riemannian manifold (M, g) is called an *Einstein manifold* if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v, w) = c\langle v, w\rangle$$

for every $p \in M$, $v, w \in T_pM$.

(a) Show that (M, g) is an Einstein manifold if and only if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v) = c$$

for every $p \in M$ and unit tangent vector $v \in T_pM$.

(b) Show that if (M, g) is of constant sectional curvature then (M, g) is an Einstein manifold.

4.3. Let (M, g) be a Riemannian manifold. The goal of this exercise is to show that M is of constant sectional curvature K_0 if and only if

$$\langle R(v_1, v_2)v_3, v_4\rangle = -K_0(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle)$$

for any $p \in M$ and $v_1, v_2, v_3, v_4 \in T_pM$. Denote the expression $-K_0\left(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_2, v_3\rangle\langle v_1, v_4\rangle\right)$ by (v_1, v_2, v_3, v_4).

(a) Show that if

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in T_pM$, then M is of constant sectional curvature K_0.

Now assume that M is of constant sectional curvature K_0. Our aim is to show that

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4)$$

for any four tangent vectors $v_1, v_2, v_3, v_4 \in T_pM$.

(b) Show that the expression (v_1, v_2, v_3, v_4) is a tensor, i.e. it is multilinear.

(c) Show that (v_1, v_2, v_3, v_4) has the same symmetries as Riemann curvature tensor has. Namely,

- $(v_1, v_2, v_3, v_4) = -(v_2, v_1, v_3, v_4)$
- $(v_1, v_2, v_3, v_4) = -(v_1, v_2, v_4, v_3)$
- $(v_1, v_2, v_3, v_4) = (v_3, v_4, v_1, v_2)$
- $(v_1, v_2, v_3, v_4) + (v_2, v_3, v_1, v_4) + (v_3, v_1, v_2, v_4) = 0$

(d) Show that if $\{v_1, v_2, v_3, v_4\} \subset \{v, w\}$, i.e. no more than two distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(e) Show that if no more than three distinct vectors are involved, then

$$\langle R(v_1, v_2)v_3, v_4\rangle = (v_1, v_2, v_3, v_4).$$

(f) Show that for any four vectors $\{v_1, v_2, v_3, v_4\}$

$$\langle R(v_1, v_2)v_3, v_4\rangle - (v_1, v_2, v_3, v_4) = \langle R(v_3, v_1)v_2, v_4\rangle - (v_3, v_1, v_2, v_4),$$

i.e. the difference above is invariant with respect to cyclic permutation of first three arguments.

(g) Use Bianchi identity to prove the initial statement.

4.4. Constant sectional curvature of hyperbolic 3-space

Let $\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$ be the upper half-space model of the 3-dimensional hyperbolic space, i.e. its metric is defined by $g_{ij} = 0$ for $i \neq j$, $g_{ii} = 1/x_3^2$.

(a) Show that sectional curvatures $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})$ and $K(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ in $\mathbb{H}^3$ are equal to -1.

(b) Use (a) and the linearity of the Riemann curvature tensor to show that for any $p \in \mathbb{H}^3$ and $v_1, v_2, v_3, v_4 \in T_p\mathbb{H}^3$

$$\langle R(v_1, v_2)v_3, v_4\rangle = -(\langle v_1, v_3\rangle\langle v_2, v_4\rangle - \langle v_1, v_4\rangle\langle v_2, v_3\rangle)$$

holds.

(c) Use (b) to show that 3-dimensional hyperbolic space $\mathbb{H}^3$ has constant sectional curvature -1.

(d) Show that n-dimensional hyperbolic space $\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}$ with metric $g_{ij} = 0$ for $i \neq j$, $g_{ii} = 1/x_n^2$ has constant sectional curvature -1.