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Riemannian Geometry IV, Solutions 5 (Week 15)

5.1. ($\star$) The Bonnet – Myers theorem claims that if (M,g) is complete and connected, and there is $\varepsilon > 0$ such that $Ric_p(v) \geq \varepsilon$ for every $p \in M$ and for every unit tangent vector v, then the diameter of M is finite.

Show by example that the assumption $\varepsilon > 0$ is essential (i.e. cannot be substituted by the assumption $Ric_p(v) > 0$).

Solution: One may consider an elliptic paraboloid of revolution $z = x^2 + y^2$. Its curvature is positive, but the paraboloid is not compact (e.g., it is unbounded). Note that although the curvature is positive (since the manifold is 2-dimensional sectional and Ricci curvatures coincide) it is not separated from zero, so there is no contradiction with Bonnet-Myers theorem.

5.2. Second Variational Formula of Energy

Let $F : (-\varepsilon, \varepsilon) \times [a,b] \to M$ be a proper variation of a geodesic $c : [a,b] \to M$, and let X be its variational vector field. Let $E : (-\varepsilon, \varepsilon) \to \mathbb{R}$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2} \int_a^b \|\frac{\partial F}{\partial t}(s,t)\|^2 dt.$$

Show that

$$E''(0) = \int_a^b \|\frac{D}{dt} X\|^2 - \langle R(X, c')c', X\rangle \, dt$$

Solution:

Since $E(s) = \frac{1}{2}\int_a^b \langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\rangle \, dt$, using the Riemannian property of covariant derivative we obtain

$$E'(s) = \int_a^b \left\langle \frac{D}{ds}\frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t) \right\rangle dt.$$

Differentiating the integrand with respect to s, using the Symmetry Lemma, and setting then $s = 0$ yields

$$E''(0) = \int_a^b \frac{d}{ds}\Big|_{s=0} \left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial s}(s,t) \right\rangle dt.$$

Applying Riemannian property of covariant derivative, Symmetry Lemma, and using that $\frac{\partial F}{\partial s}(0,t) = X(t)$, we conclude that

$$E''(0) = \int_a^b \left\langle \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(0,t) \right\rangle dt + \int_a^b \left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(0,t), \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial t}(s,t) \right\rangle dt =$$
$$= \int_a^b \left\langle \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), \frac{\partial F}{\partial t}(0,t) \right\rangle dt + \int_a^b \left\langle \frac{D}{dt}\frac{\partial F}{\partial s}(0,t), \frac{D}{dt}\frac{\partial F}{\partial s}(0,t) \right\rangle dt =$$
$$= \int_a^b \left\langle \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle dt + \int_a^b \left\| \frac{DX}{dt} \right\|^2 dt$$

Now we use Lemma 8.5 to interchange the order of covariant derivatives, and again Riemannian property to obtain

$$\left\langle \frac{D}{ds}\Big|_{s=0} \frac{D}{dt}\frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle =$$
$$= \left\langle \frac{D}{dt}\frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle + \left\langle R\left(\frac{\partial F}{\partial s}(0,t), \frac{\partial F}{\partial t}(0,t)\right) \frac{\partial F}{\partial s}(0,t), c'(t) \right\rangle =$$
$$= \frac{d}{dt}\left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle - \left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), \frac{D}{dt}c'(t) \right\rangle + \langle R(X(t), c'(t))X'(t), c'(t)\rangle =$$
$$= \frac{d}{dt}\left\langle \frac{D}{ds}\Big|_{s=0} \frac{\partial F}{\partial s}(s,t), c'(t) \right\rangle - \langle R(X(t), c'(t))c'(t), X(t)\rangle,$$

since $c(t)$ is geodesic and $\frac{D}{dt}c'(t) = 0$.

Now we are left to show that

$$\int_a^b \frac{d}{dt}\left\langle \frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial s}(s,t), c'(t)\right\rangle dt = 0$$

Indeed,

$$\int_a^b \frac{d}{dt}\left\langle \frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial s}(s,t), c'(t)\right\rangle dt = \left\langle \frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial s}(s,b), c'(b)\right\rangle - \left\langle \frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial s}(s,a), c'(a)\right\rangle,$$

but

$$\frac{\partial F}{\partial s}(s,a) = \frac{\partial F}{\partial s}(s,b) = 0$$

since the variation $F(s,t)$ is proper.

5.3. Let $S^2 = \{x \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$ be a unit sphere, and $c : [-\pi/2, \pi/2] \to S^2$ be a geodesic defined by $c(t) = (\cos t, 0, \sin t)$. Define a vector field $X : [-\pi/2, \pi/2] \to TS^2$ along c by

$$X(t) = (0, \cos t, 0).$$

Let $\frac{D}{dt}$ denote the covariant derivative along c.

(a) Calculate $\frac{D}{dt}X(t)$ and $\frac{D^2}{dt^2}X(t)$.

(b) Show that X satisfies the Jacobi equation.

Solution:

The problem can be solved by a direct computation: compute Christoffel symbols, and then compute first and second covariant derivatives of $X(t)$, then verify the Jacobi equation for $X(t)$.

(a) If we parametrize the sphere by $(x,y,z) = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, one has $\Gamma^2_{11} = -\sin\vartheta\cos\vartheta$, $\Gamma^1_{12} = \Gamma^1_{21} = \cot\vartheta$ with others Γ^k_{ij} equal to zero, where $\varphi = x_1$ and $\vartheta = x_2$ (see Exercise 3.4).
In these coordinates, the curve $c(t) = (\cos t, 0, \sin t)$ is $c(t) = (0, \frac{\pi}{2} - t)$, $c'(t) = (0, -1) = -\frac{\partial}{\partial\vartheta}$. Further, observe that

$$\frac{\partial}{\partial\varphi}\Big|_{c(t)} = (-\sin\vartheta\sin\varphi, \sin\vartheta\cos\varphi, 0)\Big|_{\varphi=0,\,\vartheta=\frac{\pi}{2}-t} = (0, \cos t, 0) = X(t)$$

Hence,

$$\frac{D}{dt}X(t) = \nabla_{c'(t)}X(t) = \nabla_{-\frac{\partial}{\partial\vartheta}}\frac{\partial}{\partial\varphi} = -\cot\vartheta\frac{\partial}{\partial\varphi}\Big|_{c(t)} = -\tan t\, X(t),$$

$$\frac{D^2}{dt^2}X(t) = \frac{D}{dt}(-\tan t\, X(t)) = -\sec^2 t\, X(t) + \tan^2 t\, X(t) = -X(t) = -\frac{\partial}{\partial\varphi}\Big|_{c(t)}$$

(b) Compute $R(X,c')c' = \nabla_X\nabla_{c'}c' - \nabla_{c'}\nabla_X c' - \nabla_{[X,c']}c'$. As $X = \frac{\partial}{\partial\varphi}$ and $c' = -\frac{\partial}{\partial\vartheta}$, we have $[X,c'] = 0$. Also,

$$\nabla_X\nabla_{c'}c' = \nabla_{\frac{\partial}{\partial\varphi}}\nabla_{-\frac{\partial}{\partial\vartheta}} - \frac{\partial}{\partial\vartheta} = \nabla_{\frac{\partial}{\partial\varphi}}0 = 0,$$

$$\nabla_{c'}\nabla_X c' = \nabla_{-\frac{\partial}{\partial\vartheta}}\nabla_{\frac{\partial}{\partial\varphi}} - \frac{\partial}{\partial\vartheta} = \nabla_{\frac{\partial}{\partial\vartheta}}(\cot\vartheta\frac{\partial}{\partial\varphi}) = -\frac{1}{\sin^2\vartheta}\frac{\partial}{\partial\varphi} + \cot\vartheta(\cot\vartheta\frac{\partial}{\partial\varphi}) = (\cot^2\vartheta - \frac{1}{\sin^2\vartheta})\frac{\partial}{\partial\varphi} = -X(t).$$

Thus, $R(X,c')c' = X(t) = \frac{\partial}{\partial\varphi}$, and (since $\frac{D^2}{dt^2}X(t) = -X(t) = -\frac{\partial}{\partial\varphi}$) Jacobi equation holds.

5.4. Jacobi fields on manifolds of constant curvature.
Let M be a Riemannian manifold of constant sectional curvature K, and $c : [0,1] \to M$ be a geodesic parametrized by arc length. Let $J : [0,1] \to TM$ be an orthogonal Jacobi field along c (i.e. $\langle J(t), c'(t)\rangle = 0$ for every $t \in [0,1]$).

(a) Show that $R(J,c')c' = KJ$.

(b) Let $Z_1, Z_2 : [0,1] \to TM$ be parallel vector fields along c with $Z_1(0) = J(0)$, $Z_2(0) = \frac{DJ}{dt}(0)$. Show that

$$J(t) = \begin{cases} \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}} Z_2(t) & \text{if } K > 0, \\ Z_1(t) + tZ_2(t) & \text{if } K = 0, \\ \cosh(t\sqrt{-K})Z_1(t) + \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} Z_2(t) & \text{if } K < 0. \end{cases}$$

Hint: Show that these fields satisfy Jacobi equation, there value and covariant derivative at $t = 0$ is the same as for $J(t)$.

Solution:

(a) We conclude from Exercise 4.3 that

$$R(v_1, v_2)v_3 = K(\langle v_2, v_3\rangle v_1 - \langle v_1, v_3\rangle v_2).$$

This implies

$$R(J, c')c' = K(\langle c', c'\rangle J - \langle J, c'\rangle c').$$

Since $\|c'\|^2 = 1$ and $J \perp c'$, we obtain

$$R(J, c')c' = KJ.$$

(b) We only consider the case $K > 0$, all other cases are similar. The vector field

$$J(t) = \cos(t\sqrt{K})Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}} Z_2(t)$$

satisfies $J(0) = Z_1(0)$ and

$$\frac{DJ}{dt}(t) = -\sqrt{K}\sin(t\sqrt{K})Z_1(t) + \cos(t\sqrt{K})Z_2(t),$$

which implies $\frac{DJ}{dt}(0) = Z_2(0)$. Obviously, we have

$$\frac{D^2 J}{dt^2}(t) = -K\cos(t\sqrt{K})Z_1(t) - \sqrt{K}\sin(t\sqrt{K})Z_2(t) = -KJ(t),$$

and therefore we obtain

$$\frac{D^2 J}{dt^2}(t) + KJ(t) = 0,$$

i.e., J satisfies the Jacobi equation.