Durham University
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Epiphany 2020

Riemannian Geometry IV, Homework 5 (Week 15)

Due date for starred problems: **Friday, February 28**.

5.1. ($\star$) The Bonnet – Myers theorem claims that if (M,g) is complete and connected, and there is $\varepsilon > 0$ such that $Ric_p(v) \geq \varepsilon$ for every $p \in M$ and for every unit tangent vector v, then the diameter of M is finite.
Show by example that the assumption $\varepsilon > 0$ is essential (i.e. cannot be substituted by the assumption $Ric_p(v) > 0$).

5.2. Second Variation Formula of Energy
Let $F : (-\varepsilon, \varepsilon) \times [a,b] \to M$ be a proper variation of a geodesic $c : [a,b] \to M$, and let X be its variational vector field. Let $E : (-\varepsilon, \varepsilon) \to \mathbb{R}$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2} \int_a^b \| \frac{\partial F}{\partial t}(s,t) \|^2 dt.$$

Show that

$$E''(0) = \int_a^b \| \frac{D}{dt} X \|^2 - \langle R(X, c')c', X \rangle dt$$

5.3. Let $S^2 = \{x \in \mathbb{R}^3 \mid {x_1}^2 + {x_2}^2 + {x_3}^2 = 1\}$ be a unit sphere, and $c : [-\pi/2, \pi/2] \to S^2$ be a geodesic defined by $c(t) = (\cos t, 0, \sin t)$. Define a vector field $X : [-\pi/2, \pi/2] \to TS^2$ along c by

$$X(t) = (0, \cos t, 0).$$

Let $\frac{D}{dt}$ denote the covariant derivative along c.

(a) Calculate $\frac{D}{dt} X(t)$ and $\frac{D^2}{dt^2} X(t)$.
(b) Show that X satisfies the Jacobi equation.

5.4. Jacobi fields on manifolds of constant curvature.
Let M be a Riemannian manifold of constant sectional curvature K, and $c : [0,1] \to M$ be a geodesic parametrized by arc length. Let $J : [0,1] \to TM$ be an orthogonal Jacobi field along c (i.e. $\langle J(t), c'(t) \rangle = 0$ for every $t \in [0,1]$).

(a) Show that $R(J, c')c' = KJ$.
(b) Let $Z_1, Z_2 : [0,1] \to TM$ be parallel vector fields along c with $Z_1(0) = J(0)$, $Z_2(0) = \frac{DJ}{dt}(0)$. Show that

$$J(t) = \begin{cases} \cos(t\sqrt{K}) Z_1(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}} Z_2(t) & \text{if } K > 0, \\ Z_1(t) + t Z_2(t) & \text{if } K = 0, \\ \cosh(t\sqrt{-K}) Z_1(t) + \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} Z_2(t) & \text{if } K < 0. \end{cases}$$

Hint: Show that these fields satisfy Jacobi equation, and there value and covariant derivative at $t = 0$ is the same as for $J(t)$.