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Riemannian Geometry IV, Solutions 6 (Week 16)

6.1. ($\star$) Choose any $r > 0$ and consider a cylinder $C \subset \mathbb{R}^3$ with induced metric,

$$C = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = r^2\}$$

C can be parametrized by

$$(r\cos\varphi, r\sin\varphi, z), \quad \varphi \in [0, 2\pi), z \in \mathbb{R}$$

(a) Show that a curve $c(t) = (r\cos(t/r), r\sin(t/r), 0)$ is a geodesic. Write $c(t)$ in the form $(\varphi(t), z(t))$.

(b) Let $\alpha \in \mathbb{R}$. Show that $c_\alpha(t) = (\varphi(t), z(t)) = ((t\cos\alpha)/r, t\sin\alpha)$ is a geodesic.

(c) Construct two distinct geodesic variations $F_1(s,t)$ and $F_2(s,t)$ of $c(t)$, such that $F_1(s, 0) \equiv c(0)$, and $F_2(s, 0) \neq c(0)$ for any $s \neq 0$. Compute the variational vector fields of F_1 and F_2.

(d) Construct the basis of the space J_c of Jacobi fields along $c(t)$.

(e) Show that for any $t_0 \in \mathbb{R}$ the points $c(0)$ and $c(t_0)$ are not conjugate along $c(t)$.

Solution:

(a) One way to do this is to use symmetry of C. More precisely, the reflection in the plane $z = 0$ is obviously an isometry of C, and it preserves $c(t)$. By the uniqueness theorem of a geodesic in a given direction, the trace of $c(t)$ should be a trace of a geodesic. Now observe that $\|c'(t)\| = 1$, so $c(t)$ is a geodesic.
Another way is to observe that C is locally isometric to $\mathbb{R}^2$, and the isometry takes $c(t)$ to a straight line on $\mathbb{R}^2$ (parametrized proportionally to arc length).
Finally, one can compute the induced metric and Christoffel symbols (they are all zeros!), and then verify that $c(t)$ satisfies the ODE for geodesics. These ODE's are then equivalent to the second derivatives of the components of $c(t)$ vanishing identically, which is clearly satisfied as the geodesic $c(t)$ is written as $c(t) = (t/r, 0)$ in coordinates (φ, z).

(b) The second and the third methods from (a) work perfectly fine in this case as well.

(c) We can take

$$F_1(s,t) = \left(r\cos\left(\frac{t\cos s}{r}\right), r\sin\left(\frac{t\cos s}{r}\right), t\sin s\right) = c_s(t)$$

Clearly, $F_1(0,t) = c(t)$, $F_1(s, 0) \equiv (r, 0, 0) = c(0)$, and every $t \mapsto F_1(s_0, t)$ is a geodesic by (b). The variational vector field is $X_1(t) = (0, 0, t)$.
Shifting $c(t)$ in vertical direction, we can take

$$F_2(s,t) = (r\cos(t/r), r\sin(t/r), s)$$

The corresponding variational vector field is $X_2(t) = (0, 0, 1)$.

(d) We need $2n = 4$ linearly independent vector fields. We have already found two, and observe that X_1 and X_2 are both orthogonal and clearly linear independent, so they form a basis of the space of orthogonal Jacobi fields. We can also take $X_3(t) = c'(t)$ and $X_4 = tc'(t)$, all of them together form a basis.

(e) Assume that $J(0) = J(t_0) = 0$ for some $J \in J_c$. Since $J(0) = 0$, J should be a linear combination of X_1 and X_4. However, such a non-zero linear combination never vanishes except for $t = 0$.

6.2. ($\star$) Let $c : [0, 1] \to M$ be a geodesic, and let J be a Jacobi field along c. Denote $c(0) = p, c'(0) = v$. Define a curve $\gamma(s)$,

$$\gamma : (-\varepsilon, \varepsilon) \to M, \qquad \gamma(0) = p, \gamma'(0) = J(0)$$

Define also a vector field $V(s) \in \mathfrak{X}_\gamma(M)$ such that

$$V(0) = v, \qquad \frac{D}{ds}V(0) = \frac{D}{dt}J(0),$$

and a variation $F(s,t) = \exp_{\gamma(s)} tV(s)$.

(a) Show that $F(s,t)$ is a geodesic variation of $c(t)$.

(b) Show that $\frac{\partial F}{\partial s}(0,0) = \gamma'(0) = J(0)$, and $\frac{D}{dt}\frac{\partial F}{\partial s}(0,0) = \frac{D}{ds}V(0) = \frac{D}{dt}J(0)$.

(c) Deduce from (a) and (b) that every Jacobi field along a geodesic $c(t)$ is a variational vector field of some geodesic variation of c.

Solution:

(a) By the definition of the exponential map, for given s the curve $t \mapsto \exp_{\gamma(s)} tV(s)$ is a geodesic.

(b) We have

$$\frac{\partial F}{\partial s}(0,0) = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\gamma(s)} tV(s)\Big|_{t=0} = \frac{\partial}{\partial s}\Big|_{s=0} \exp_{\gamma(s)}(0) = \frac{\partial}{\partial s}\Big|_{s=0}\gamma(s) = \gamma'(0) = J(0),$$

and

$$\frac{D}{dt}\frac{\partial F}{\partial s}(0,0) = \frac{D}{ds}\Big|_{s=0}\frac{\partial F}{\partial t}\Big|_{t=0}(s,t) = \frac{D}{ds}\Big|_{s=0} V(s) = \frac{D}{ds}V(0) = \frac{D}{dt}J(0)$$

(c) According to (a), the variation F is geodesic, thus its variational vector field $\frac{\partial F}{\partial s}(0,t)$ is Jacobi. By (b), $\frac{\partial F}{\partial s}(0,0) = J(0)$, and $\frac{D}{dt}\frac{\partial F}{\partial s}(0,0) = \frac{D}{dt}J(0)$, which means that $\frac{\partial F}{\partial s}(0,t) = J(t)$ due to the uniqueness theorem.

6.3. Jacobi fields and conjugate points on locally symmetric spaces

A Riemannian manifold (M,g) is called a *locally symmetric space* if $\nabla R = 0$ (see Exercise 9.3). Let (M,g) be an n-dimensional locally symmetric space and $c : [0,\infty) \to M$ be a geodesic with $p = c(0)$ and $v = c'(0) \in T_pM$. Prove the following facts:

(a) Let X, Y, Z be parallel vector fields along c. Show that $R(X,Y)Z$ is also parallel.

(b) Let $K_v \in \mathrm{Hom}\,(T_pM, T_pM)$ be the curvature operator defined by

$$K_v(w) = R(w,v)v.$$

Show that K_v is self-adjoint, i.e.,

$$\langle K_v(w_1), w_2\rangle = \langle w_1, K_v(w_2)\rangle$$

for every pair of vectors $w_1, w_2 \in T_pM$.

(c) Choose an orthonormal basis $w_1, \dots, w_n \in T_pM$ that diagonalizes K_v, i.e.,

$$K_v(w_i) = \lambda_i w_i$$

(such a basis exists since K_v is self-adjoint). Let $W_1, \dots, W_n$ be the parallel vector fields along c with $W_i(0) = w_i$ (i.e., $\{W_i\}$ form a parallel orthonormal basis along c). Show that for all $t \in [0,\infty)$

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t).$$

(d) Let $J(t) = \sum_i J_i(t)W_i(t)$ be a Jacobi field along c. Show that Jacobi equation translates into

$$J_i''(t) + \lambda_i J_i(t) = 0, \quad \text{for } i = 1, \dots, n.$$

(e) Show that the conjugate points of p along c are given by $c(\pi k/\sqrt{\lambda_i})$, where k is any positive integer and λ_i is a positive eigenvalue of K_v.

Solution:

(a) We know that $\nabla R = 0$. Let $\frac{D}{dt}$ denote covariant derivative along c. Then we have, for parallel vector fields X, Y, Z along c that

$$
\begin{aligned}
0 = \nabla R(X,Y,Z,c')(t) &= \frac{D}{dt} R(X(t),Y(t))Z(t) - \\
&- R(\underbrace{\frac{D}{dt}X(t)}_{=0},Y(t))Z(t) - R(X(t),\underbrace{\frac{D}{dt}Y(t)}_{=0})Z(t) - R(X(t),Y(t))\underbrace{\frac{D}{dt}Z(t)}_{=0} = \\
&= \frac{D}{dt} R(X(t),Y(t))Z(t).
\end{aligned}
$$

This shows that $R(X,Y)Z$ is parallel.

(b) The symmetries of R yield

$$\langle K_v(w_1), w_2\rangle = \langle R(w_1,v)v, w_2\rangle = \langle R(w_2,v)v, w_1\rangle = \langle K_v(w_2), w_1\rangle = \langle w_1, K_v(w_2)\rangle.$$

(c) Since K_v is self-adjoint, we can find an orthonormal basis $w_1, \dots, w_n \in T_pM$ with $K_v(w_i) = \lambda_i w_i$. We know, by (a), that $K_{c'(t)}(W_i(t)) = R(c'(t), W_i(t))W_i(t)$ is parallel and, since $K_{c'(0)}(W_i(0)) = K_v(w_i) = \lambda_i w_i$, we must have

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t),$$

since parallel vector fields V along c are uniquely determined by their initial values $V(0) \in T_pM$.

(d) Let J be a Jacobi field along c. Then J satisfies the Jacobi equation

$$\frac{D^2}{dt^2} J + R(J,c')c' = 0.$$

Since $W_1, \dots, W_n$ form a parallel basis along c, we obtain, by taking inner product with W_i:

$$
\begin{aligned}
0 = \langle \frac{D^2}{dt^2} J, W_i\rangle + \langle R(J,c')c', W_i\rangle &= \\
&= \frac{d^2}{dt^2} \sum_j J_j \langle W_j, W_i\rangle + \sum_j J_j \langle R(W_j,c')c', W_i\rangle = \\
&= J_i'' + \sum_j J_j \lambda_j \langle W_j, W_i\rangle = J_i'' + \lambda_i J_i.
\end{aligned}
$$

(e) The unique solution of $J_i''(t) + \lambda_i J_i(t) = 0$, $J_i(0) = 0$ (up to scalar multiples) is given by

$$J_i(t) = \begin{cases} \sin(t\sqrt{\lambda_i}) & \text{if } \lambda_i > 0, \\ t & \text{if } \lambda_i = 0, \\ \sinh(t\sqrt{-\lambda_i}) & \text{if } \lambda_i < 0. \end{cases}$$

So J_i has zeros for positive t only if $\lambda_i > 0$, and these are precisely at $t = \pi k/\sqrt{\lambda_i}$. The corresponding Jacobi fields with $J(0) = 0$ and $\frac{D}{dt}J(0) = w_i$ produce the conjugate points $c(\pi k/\sqrt{\lambda_i})$.