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Epiphany 2020

Riemannian Geometry IV, Homework 6 (Week 16)

Due date for starred problems: **Friday, February 28**.

6.1. ($\star$) Choose any $r > 0$ and consider a cylinder $C \subset \mathbb{R}^3$ with induced metric,

$$C = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = r^2\}$$

C can be parametrized by

$$(r\cos\varphi, r\sin\varphi, z), \quad \varphi \in [0, 2\pi), z \in \mathbb{R}$$

(a) Show that a curve $c(t) = (r\cos(t/r), r\sin(t/r), 0)$ is a geodesic. Write $c(t)$ in the form $(\varphi(t), z(t))$.

(b) Let $\alpha \in \mathbb{R}$. Show that $c_\alpha(t) = (\varphi(t), z(t)) = ((t\cos\alpha)/r, t\sin\alpha)$ is a geodesic.

(c) Construct two distinct geodesic variations $F_1(s, t)$ and $F_2(s, t)$ of $c(t)$, such that $F_1(s, 0) \equiv c(0)$, and $F_2(s, 0) \neq c(0)$ for any $s \neq 0$. Compute the variational vector fields of F_1 and F_2.

(d) Construct the basis of the space J_c of Jacobi fields along $c(t)$.

(e) Show that for any $t_0 \in \mathbb{R}$ the points $c(0)$ and $c(t_0)$ are not conjugate along $c(t)$.

6.2. ($\star$) Let $c : [0, 1] \to M$ be a geodesic, and let J be a Jacobi field along c. Denote $c(0) = p, c'(0) = v$. Define a curve $\gamma(s)$,

$$\gamma : (-\varepsilon, \varepsilon) \to M, \qquad \gamma(0) = p, \gamma'(0) = J(0)$$

Define also a vector field $V(s) \in \mathfrak{X}_\gamma(M)$, such that

$$V(0) = v, \qquad \frac{D}{ds}V(0) = \frac{D}{dt}J(0),$$

and a variation $F(s, t) = exp_{\gamma(s)} tV(s)$.

(a) Show that $F(s, t)$ is a geodesic variation of $c(t)$.

(b) Show that $\frac{\partial F}{\partial s}(0, 0) = \gamma'(0) = J(0)$, and $\frac{D}{dt}\frac{\partial F}{\partial s}(0, 0) = \frac{D}{ds}V(0) = \frac{D}{dt}J(0)$.

(c) Deduce from (a) and (b) that every Jacobi field along a geodesic $c(t)$ is a variational vector field of some geodesic variation of c.

6.3. Jacobi fields and conjugate points on locally symmetric spaces
A Riemannian manifold (M, g) is called a *locally symmetric space* if $\nabla R = 0$ (see Exercise 9.3). Let (M, g) be an n-dimensional locally symmetric space and $c : [0, \infty) \to M$ be a geodesic with $p = c(0)$ and $v = c'(0) \in T_pM$. Prove the following facts:

(a) Let X, Y, Z be parallel vector fields along c. Show that $R(X, Y)Z$ is also parallel.

(b) Let $K_v \in \mathrm{Hom}\,(T_pM, T_pM)$ be the curvature operator defined by

$$K_v(w) = R(w, v)v.$$

Show that K_v is self-adjoint, i.e.,

$$\langle K_v(w_1), w_2\rangle = \langle w_1, K_v(w_2)\rangle$$

for every pair of vectors $w_1, w_2 \in T_pM$.

(c) Choose an orthonormal basis $w_1, \ldots, w_n \in T_pM$ that diagonalizes K_v, i.e.,

$$K_v(w_i) = \lambda_i w_i$$

(such a basis exists since K_v is self-adjoint). Let $W_1, \ldots, W_n$ be the parallel vector fields along c with $W_i(0) = w_i$ (i.e., $\{W_i\}$ form a parallel orthonormal basis along c). Show that for all $t \in [0, \infty)$

$$K_{c'(t)}(W_i(t)) = \lambda_i W_i(t).$$

(d) Let $J(t) = \sum_i J_i(t) W_i(t)$ be a Jacobi field along c. Show that Jacobi equation translates into

$$J_i''(t) + \lambda_i J_i(t) = 0, \quad \text{for } i = 1, \ldots, n.$$

(e) Show that the conjugate points of p along c are given by $c(\pi k / \sqrt{\lambda_i})$, where k is any positive integer and λ_i is a positive eigenvalue of K_v.