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Riemannian Geometry IV, Solutions 7 (Week 17)

7.1. (a) Let $c(t)$ be a geodesic, and let $c(t_0)$ be conjugate to $c(t_1)$. Let J be any Jacobi field along c vanishing at t_0 and t_1. Show that J is orthogonal, i.e. $\langle J(t), c'(t)\rangle \equiv 0$.

(b) Show that the dimension of the space $J_c^\perp$ of orthogonal vector fields along c is $2n-2$.

Solution:

(a) We have proved in class that the function $t \mapsto \langle J(t), c'(t)\rangle$ is linear. Since it is equal to zero at two points t_0 and t_1, it vanishes everywhere, so $J(t)$ is orthogonal.

(b) Recall that $J(t)$ is orthogonal if and only if both $\langle J(0), c'(0)\rangle$ and $\langle \frac{D}{dt}J(0), c'(0)\rangle$ vanish. Each of these equations defines a codimension one subspace in T_pM, so the dimension of $J_c^\perp = (n-1) + (n-1) = 2n-2$.

7.2. ($\star$) Let M be a Riemannian manifold of non-positive sectional curvature, i.e. $K(\Pi) \le 0$ for any 2-plane $\Pi \subset TM$.

(a) Let $c : [a, b] \to M$ be a geodesic and let J be a Jacobi field along c. Let $f(t) = \|J(t)\|^2$. Show that $f''(t) \ge 0$, i.e., f is a convex function.

(b) Derive from (a) that M does not admit conjugate points.

Solution:

(a) We have

$$f'(t) = \frac{d}{dt}\Big|_{t=0} \langle J(t), J(t)\rangle = 2\langle \frac{D}{dt}J(t), J(t)\rangle$$

and

$$f''(t) = 2\left(\langle \frac{D^2}{dt^2}J(t), J(t)\rangle + \left\|\frac{D}{dt}J(t)\right\|^2\right).$$

Using Jacobi equation, we conclude

$$f''(t) = 2\left(-\langle R(J(t), c'(t))c'(t), J(t)\rangle + \left\|\frac{D}{dt}J(t)\right\|^2\right).$$

We have $\langle R(J(t), c'(t))c'(t), J(t)\rangle = 0$ if $J(t), c'(t)$ are linear dependent and, otherwise, for $\Pi = \mathrm{span}(J(t), c'(t)) \subset T_{c(t)}M$,

$$\langle R(J(t), c'(t))c'(t), J(t)\rangle = K(\Pi)\left(\|J(t)\|^2\|c'(t)\|^2 - (\langle J(t), c'(t)\rangle)^2\right) \le 0,$$

since sectional curvature is non-positive (we also used Cauchy – Schwarz inequality here). This shows that $f''(t)$, as a sum of two non-negative terms, is greater than or equal to zero.

(b) If there were a conjugate point $q = c(t_2)$ to a point $p = c(t_1)$ along the geodesic c, then we would have a non-vanishing Jacobi field J along c with $J(t_1) = 0$ and $J(t_2) = 0$. This would imply that the convex, non-negative function $f(t) = \|J(t)\|^2$ would have zeros at $t = t_1$ and $t = t_2$. This would force f to vanish identically on the interval $[t_1, t_2]$, which would imply that J vanishes as well, which leads to a contradiction.

7.3. ($\star$) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}$ be a paraboloid of revolution with metric induced by $\mathbb{R}^3$. Let $p = (0, 0, 0)$. Show that p has no conjugate points in M.

Solution:

Let $q = (q_1, q_2, q_3) \neq p$ be any point in M. Denote by $\Pi \subset \mathbb{R}^3$ the 2-dimensional plane spanned by q and the z-axis. It is easy to check that there is a geodesic $c(t) \subset M \cap \Pi$ with $c(0) = p$, $c(t_1) = q$.

Take a parallel of M passing through q. Applying the same argument to all other points on the parallel, we see that every "vertical" plane through p in $\mathbb{R}^3$ contains a geodesic of M, and only one of them (namely, c) passes through q. As these geodesics exhaust all the directions from p, we conclude that c is the only geodesic between p and q, and thus it is minimal. By Theorem 9.24 this implies that for any $t_0 \in (0, t_1)$ the point $c(t_0)$ is not conjugate to p.

Rotating the whole picture around the z-axis (this is clearly an isometry of M) we see that p has no conjugate points in a ball $z < q_3$ (where $q = (q_1, q_2, q_3)$), so taking q far enough from p we can prove that p has no conjugate points in a ball of any size centered at p.

7.4. Let (M, g) be a Riemannian manifold. For a tensor T let ∇T denote its covariant derivative, see Exercise 9.3. T is called a *parallel tensor* if $\nabla T = 0$.

(a) Assume that $T_1, T_2 : \mathfrak{X} \times \mathfrak{X} \to C^\infty(M)$ are parallel tensors. Show that the tensor $T : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to C^\infty(M)$, defined as

$$T(X_1, X_2, X_3, X_4) = T_1(X_1, X_2) T_2(X_3, X_4),$$

is also parallel.

(b) Use (a) to show that $\nabla R' = 0$ for the tensor

$$R'(X, Y, Z, W) = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle.$$

(c) Use Exercise 4.3 and (b) to show that all manifolds with constant sectional curvature have parallel Riemann curvature tensor

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W\rangle.$$

Solution:

(a) We have

$$\begin{aligned}
&\nabla T(X_1, X_2, X_3, X_4, Y) = \\
&\quad = Y(T_1(X_1, X_2)T_2(X_3, X_4)) - \sum_{i=1}^{4} T(X_1, \dots, \nabla_Y X_i, \dots, X_4) = \\
&\quad = T_1(X_1, X_2)\underbrace{\left(Y(T_2(X_3, X_4)) - T_2(\nabla_Y X_3) - T_2(\nabla_Y X_4)\right)}_{=\nabla T_2(X_3, X_4, Y) = 0} + \\
&\qquad + T_2(X_3, X_4)\underbrace{\left(Y(T_1(X_1, X_2)) - T_1(\nabla_Y X_1) - T_1(\nabla_Y X_2)\right)}_{=\nabla T_1(X_1, X_2, Y) = 0} = 0.
\end{aligned}$$

(b) Let $T(X, Y) = \langle X, Y\rangle$. Since ∇ is Riemannian, we have

$$\nabla T(X, Y, Z) = Z(\langle X, Y\rangle) - \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle = 0.$$

Note that $R'(X, Y, Z, W) = T(X, W)T(Y, Z) - T(X, Z)T(Y, W)$. Part (a) implies then that we have $\nabla R' = 0$.

(c) If (M, g) is a manifold with constant sectional curvature $K_0 \in \mathbb{R}$, we have by Exercise 4.3

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle = K_0(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle) = K_0 R'(X,Y,Z,W).$$

Then $\nabla R = K_0 \nabla R' = 0$ follows from (b).