Durham University
Pavel Tumarkin

Epiphany 2020

Riemannian Geometry IV, Homework 7 (Week 17)

Due date for starred problems: **Friday, March 13**.

7.1. (a) Let $c(t)$ be a geodesic, and let $c(t_0)$ be conjugate to $c(t_1)$. Let J be any Jacobi field along c vanishing at t_0 and t_1. Show that J is orthogonal, i.e. $\langle J(t), c'(t)\rangle \equiv 0$.

(b) Show that the dimension of the space $J_c^{\perp}$ of orthogonal vector fields along c is $2n-2$.

7.2. ($\star$) Let M be a Riemannian manifold of non-positive sectional curvature, i.e. $K(\Pi) \leq 0$ for any 2-plane $\Pi \subset TM$.

(a) Let $c : [a, b] \to M$ be a geodesic and let J be a Jacobi field along c. Let $f(t) = \|J(t)\|^2$. Show that $f''(t) \geq 0$, i.e., f is a convex function.

(b) Derive from (a) that M does not admit conjugate points.

7.3. ($\star$) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}$ be a paraboloid of revolution with metric induced by $\mathbb{R}^3$. Let $p = (0, 0, 0)$. Show that p has no conjugate points in M.

7.4. Let (M, g) be a Riemannian manifold. For a tensor T let ∇T denote its covariant derivative, see Exercise 9.3. T is called a *parallel tensor* if $\nabla T = 0$.

(a) Assume that $T_1, T_2 : \mathfrak{X} \times \mathfrak{X} \to C^\infty(M)$ are parallel tensors. Show that the tensor $T : \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to C^\infty(M)$, defined as

$$T(X_1, X_2, X_3, X_4) = T_1(X_1, X_2)T_2(X_3, X_4),$$

is also parallel.

(b) Use (a) to show that $\nabla R' = 0$ for the tensor

$$R'(X, Y, Z, W) = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle.$$

(c) Use Exercise 4.3 and (b) to show that all manifolds with constant sectional curvature have parallel Riemann curvature tensor

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W\rangle.$$