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Riemannian Geometry IV, Solutions 8 (Week 18)

8.1. Let (M, g) be a Riemannian manifold and $v_1, \dots, v_n \in T_pM$ be an orthonormal basis. We know from Exercise 10.4 for the geodesic normal coordinates $\varphi : B_\epsilon(p) \to B_\epsilon(0) \subset \mathbb{R}^n$,

$$\varphi^{-1}(x_1, \dots, x_n) = \exp_p(\sum x_i v_i)$$

that $\frac{\partial}{\partial x_i}|_p = v_i$ and $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$. Define an *orthonormal frame* $E_1, \dots, E_n : B_\epsilon(p) \to TM$ by Gram − Schmidt orthonormalization, i.e.,

$$\begin{aligned} F_1(q) &:= \left.\frac{\partial}{\partial x_1}\right|_q, \qquad E_1(q) := \frac{1}{\|F_1(q)\|} F_1(q), \\ &\vdots \\ F_k(q) &:= \left.\frac{\partial}{\partial x_k}\right|_q - \sum_{j=1}^{k-1} \left\langle \left.\frac{\partial}{\partial x_k}\right|_q, E_j(q) \right\rangle E_j(q), \qquad E_k(q) := \frac{1}{\|F_k(q)\|} F_k(q), \\ &\vdots \end{aligned}$$

By construction, we have $E_i(p) = v_i$ and $E_1(q), \dots, E_n(q)$ are orthonormal in T_qM for all $q \in B_\epsilon(p)$.

(a) Prove by induction on k that

$$\begin{aligned} \left(\nabla_{\frac{\partial}{\partial x_i}} F_k\right)(p) &= 0, \\ \nabla_{\frac{\partial}{\partial x_i}} \langle F_k, F_k \rangle^{-1/2}(p) &= 0, \\ \left(\nabla_{\frac{\partial}{\partial x_i}} E_k\right)(p) &= 0, \end{aligned}$$

for all $i \in \{1, \dots, n\}$.

(b) Show that

$$(\nabla_{E_i} E_j)(p) = 0$$

for all $i, j \in \{1, \dots, n\}$.

Solution:

(a) Induction proof for

$$\left(\nabla_{\frac{\partial}{\partial x_i}} F_k\right)(p) = 0, \tag{1}$$

$$\nabla_{\frac{\partial}{\partial x_i}} \langle F_k, F_k \rangle^{-1/2}(p) = 0, \tag{2}$$

$$\left(\nabla_{\frac{\partial}{\partial x_i}} E_k\right)(p) = 0, \tag{3}$$

for all $i \in \{1, \ldots, n\}$.

One easily checks (1), (2), (3) for $k = 1$. Assume all three equations hold for k. Then we obtain

$$\left(\nabla_{\frac{\partial}{\partial x_i}} F_{k+1}\right)(p) = \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_{k+1}}\right)(p) - \frac{\partial}{\partial x_i}\Big|_p \sum_{j=1}^{k} \left\langle \frac{\partial}{\partial x_{k+1}}, E_j \right\rangle E_j.$$

Using at the right hand side the product rule, the Riemannian property of the Levi-Civita connection, and the induction hypothesis $\nabla_{\frac{\partial}{\partial x_i}} E_j(p) = 0$ for $1 \le j \le k$, we conclude that the whole expression vanishes. Next, we obtain

$$\nabla_{\frac{\partial}{\partial x_i}} \langle F_{k+1}, F_{k+1}\rangle^{-1/2}(p) = -\frac{1}{\|F_{k+1}(p)\|^3} \langle \nabla_{\frac{\partial}{\partial x_i}} F_{k+1}, F_{k+1}\rangle(p),$$

which implies that also this expression vanishes because of (1). Finally,

$$\left(\nabla_{\frac{\partial}{\partial x_i}} E_{k+1}\right)(p) = \nabla_{\frac{\partial}{\partial x_i}} \langle F_{k+1}, F_{k+1}\rangle^{-1/2}(p) F_{k+1}(p) + \frac{1}{\|F_{k+1}(p)\|} \left(\nabla_{\frac{\partial}{\partial x_i}} F_{k+1}\right)(p),$$

which vanishes again because of (1) and (2). This finishes the induction procedure.

(b) We conclude

$$\left(\nabla_{E_i} E_j\right)(p) = \nabla_{E_i(p)} E_j = 0$$

from (3), since $E_i(p)$ is just a linear combination of the basis vectors $\frac{\partial}{\partial x_k}$.

8.2. Second Bianchi Identity

Let (M, g) be a Riemannian manifold and R be the curvature tensor, defined by

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle.$$

(a) Let $E_1, \ldots, E_n : B_\epsilon(p) \to TM$ be the orthonormal frame introduced in Exercise 8.1. For simplicity, let $e_i = E_i(p)$ and $E_{ij} = [E_i, E_j]$. Show that

$$\nabla R(e_i, e_j, e_k, e_l, e_m) = \langle \nabla_{e_m} \nabla_{E_k} \nabla_{E_l} E_i - \nabla_{e_m} \nabla_{E_l} \nabla_{E_k} E_i - \nabla_{e_m} \nabla_{E_{kl}} E_i, e_j\rangle.$$

(b) Using (a) and the Riemannian curvature tensor, derive

$$\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) = \langle \nabla_{[E_{mk}, E_l] + [E_{kl}, E_m] + [E_{lm}, E_k]} E_i, e_j\rangle$$

(c) Use Jacobi identity and linearity to prove the *Second Bianchi Identity*:

$$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W) = 0,$$

for X, Y, Z, W, T vector fields on M.

Solution:

(a) Note that $E_{rs}(p) = \nabla_{e_r} E_s - \nabla_{e_s} E_r = 0$. Therefore,

$$\begin{aligned}\nabla R(e_i, e_j, e_k, e_l, e_m) &= e_m(\langle R(E_i, E_j)E_k, E_l\rangle) = e_m(\langle R(E_k, E_l)E_i, E_j\rangle) \\ &= \langle \nabla_{e_m} \nabla_{E_k} \nabla_{E_l} E_i - \nabla_{e_m} \nabla_{E_l} \nabla_{E_k} E_i - \nabla_{e_m} \nabla_{E_{kl}} E_i, e_j\rangle.\end{aligned}$$

(b) (a) implies that

$$\begin{aligned}
&\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) \\
&\quad = \langle \nabla_{e_m}\nabla_{E_k}\nabla_{E_l}E_i + \nabla_{e_k}\nabla_{E_l}\nabla_{E_m}E_i + \nabla_{e_l}\nabla_{E_m}\nabla_{E_k}E_i \\
&\quad - \nabla_{e_m}\nabla_{E_l}\nabla_{E_k}E_i - \nabla_{e_l}\nabla_{E_k}\nabla_{E_m}E_i - \nabla_{e_k}\nabla_{E_m}\nabla_{E_l}E_i \\
&\quad - \nabla_{e_m}\nabla_{E_{kl}}E_i - \nabla_{e_k}\nabla_{E_{lm}}E_i - \nabla_{e_l}\nabla_{E_{mk}}E_i, e_j\rangle \\
&\quad = \langle R(e_m, e_k, \nabla_{e_l}E_i) + \nabla_{E_{mk}(p)}\nabla_{E_l}E_i - \nabla_{e_l}\nabla_{E_{mk}}E_i \\
&\quad + R(e_k, e_l, \nabla_{e_m}E_i) + \nabla_{E_{kl}(p)}\nabla_{E_m}E_i - \nabla_{e_m}\nabla_{E_{kl}}E_i \\
&\quad + R(e_l, e_m, \nabla_{e_k}E_i) + \nabla_{E_{lm}(p)}\nabla_{E_k}E_i - \nabla_{e_k}\nabla_{E_{lm}}E_i, e_j\rangle.
\end{aligned}$$

Using $\nabla_{e_r}E_s = 0$, all above curvature terms vanish and this result simplifies to

$$\begin{aligned}
&\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) \\
&\quad = \langle R(E_{mk}(p), e_l, e_i) + \nabla_{[E_{mk},E_l]}E_i + R(E_{kl}(p), e_m, e_i) + \nabla_{[E_{kl},E_m]}E_i \\
&\quad + R(E_{lm}(p), e_k, e_i) + \nabla_{[E_{lm},E_k]}E_i, e_j\rangle.
\end{aligned}$$

Using $E_{rs}(p) = 0$, this simplifies further to

$$\begin{aligned}
&\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) \\
&\quad = \langle \nabla_{[E_{mk},E_l]+[E_{kl},E_m]+[E_{lm},E_k]}E_i, e_j\rangle.
\end{aligned}$$

(c) Jacobi identity tell us that $[E_{mk}, E_l] + [E_{kl}, E_m] + [E_{lm}, E_k] = 0$, and therefore we obtain

$$\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) = 0.$$

Since this holds for any choice of basis vectors in every slot, we obtain the same result for any choice of arbitrary tangent vectors in T_pM in each slot, by linearity.

8.3. Schur Theorem

Let (M, g) be a connected Riemannian manifold of dimension $n \geq 3$ with the following property: there is a function $f : M \to \mathbb{R}$ such that, for every $p \in M$, the sectional curvature of **all** 2-planes $\Pi \subset T_pM$ satisfies

$$K(\Sigma) = f(p).$$

(a) Define $R(X, Y, Z, W) = \langle R(X, Y)Z, W\rangle$ and

$$R'(X, Y, Z, W) = \langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle.$$

Use Exercises 4.3 and 7.4 to show that $\nabla R(X, Y, Z, W, U) = (Uf)R'(X, Y, Z, W)$ (for the definition of the covariant derivative of a tensor, see Exercise 9.3).

(b) Use the Second Bianchi Identity (see Exercise 8.2) to show that

$$\begin{aligned}
&(Tf)(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle) \\
&\quad + (Zf)(\langle X, T\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, T\rangle) \\
&\quad + (Wf)(\langle X, Z\rangle\langle Y, T\rangle - \langle X, T\rangle\langle Y, Z\rangle) = 0.
\end{aligned}$$

(c) Fix a point $p \in M$ and choose $X(p), Z(p) \in T_PM$ arbitrary. Because $n \geq 3$, we can choose W, Y such that

$$\langle Z(p), W(p)\rangle_p = \langle Z(p), Y(p)\rangle_p = \langle Y(p), W(p)\rangle_p = 0,$$

and $\|Y(p)\| = 1$. Choose $T = Y$. Show that this choice yields

$$\langle (Wf)(p)Z(p) - (Zf)(p)W(p), X(p)\rangle(p) = 0,$$

and conclude that we have $(Zf)(p) = 0$.

(d) Prove *Schur Theorem*: show that f is a constant function, i.e., there is a $C \in \mathbb{R}$ such that $f(p) = C$ for all $p \in M$.

Solution:

(a) We know from Exercise 7.4(b) that the tensor R' is parallel, i.e., $\nabla R' = 0$. We conclude from (the proof of) Exercise 4.3 that $R = fR'$, and therefore

$$\nabla R(X, Y, Z, W, U) = (Uf)R'(X, Y, Z, W).$$

(b) The Second Bianchi Identity tells us that

$$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W) = 0,$$

which yields, using the definition of R':

$$\begin{aligned} 0 = (Tf)(\langle X, W\rangle\langle Y, Z\rangle - \langle X, Z\rangle\langle Y, W\rangle) \\ + (Zf)(\langle X, T\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, T\rangle) \\ + (Wf)(\langle X, Z\rangle\langle Y, T\rangle - \langle X, T\rangle\langle Y, Z\rangle). \end{aligned}$$

(c) Using the relations $\langle Z(p), W(p)\rangle = \langle Z(p), Y(p)\rangle = \langle Y(p), W(p)\rangle = 0$, $\|Y(p)\| = 1$ and $T = Y$, we conclude that, at p

$$\begin{aligned} 0 = (Tf)(p)(\langle X(p), W(p)\rangle \cdot 0 - \langle X(p), Z(p)\rangle \cdot 0) \\ + (Zf)(p)(\langle X(p), T(p)\rangle \cdot 0 - \langle X(p), W(p)\rangle \cdot 1) \\ + (Wf)(p)(\langle X(p), Z(p)\rangle \cdot 1 - \langle X(p), T(p)\rangle \cdot 0) \\ = \langle (Wf)(p)Z(p) - (Zf)(p)W(p), X(p)\rangle. \end{aligned}$$

(d) Since $Z(p)$ and $W(p)$ are linearly independent and $X(p) \in T_P M$ was arbitrary, we conclude that both $(Wf)(p) = 0$ and $(Zf)(p) = 0$. SInce $Z(p)$ was arbitrary, f must be locally constant. Since M is connected, f is globally constant.