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# Riemannian Geometry IV, Homework 8 (Week 18)

**8.1.** Let $(M, g)$ be a Riemannian manifold and $v_1, \dots, v_n \in T_pM$ be an orthonormal basis. We know from Exercise 10.4 for the geodesic normal coordinates $\varphi : B_\epsilon(p) \to B_\epsilon(0) \subset \mathbb{R}^n$,

$$\varphi^{-1}(x_1, \dots, x_n) = \exp_p(\sum x_i v_i)$$

that $\frac{\partial}{\partial x_i}|_p = v_i$ and $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$. Define an *orthonormal frame* $E_1, \dots, E_n : B_\epsilon(p) \to TM$ by Gram – Schmidt orthonormalization, i.e.,

$$\begin{aligned} F_1(q) &:= \frac{\partial}{\partial x_1}\Big|_q, \qquad E_1(q) := \frac{1}{\|F_1(q)\|} F_1(q), \\ &\vdots \\ F_k(q) &:= \frac{\partial}{\partial x_k}\Big|_q - \sum_{j=1}^{k-1} \left\langle \frac{\partial}{\partial x_k}\Big|_q, E_j(q) \right\rangle E_j(q), \qquad E_k(q) := \frac{1}{\|F_k(q)\|} F_k(q), \\ &\vdots \end{aligned}$$

By construction, we have $E_i(p) = v_i$ and $E_1(q), \dots, E_n(q)$ are orthonormal in $T_qM$ for all $q \in B_\epsilon(p)$.

(a) Prove by induction on $k$ that

$$\begin{aligned} \left(\nabla_{\frac{\partial}{\partial x_i}} F_k\right)(p) &= 0, \\ \nabla_{\frac{\partial}{\partial x_i}} \langle F_k, F_k \rangle^{-1/2}(p) &= 0, \\ \left(\nabla_{\frac{\partial}{\partial x_i}} E_k\right)(p) &= 0, \end{aligned}$$

for all $i \in \{1, \dots, n\}$.

(b) Show that

$$\left(\nabla_{E_i} E_j\right)(p) = 0$$

for all $i, j \in \{1, \dots, n\}$.

**8.2. Second Bianchi Identity**
Let $(M, g)$ be a Riemannian manifold and $R$ be the curvature tensor, defined by

$$R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle.$$

(a) Let $E_1, \dots, E_n : B_\epsilon(p) \to TM$ be the orthonormal frame introduced in Exercise 8.1. For simplicity, let $e_i = E_i(p)$ and $E_{ij} = [E_i, E_j]$. Show that

$$\nabla R(e_i, e_j, e_k, e_l, e_m) = \langle \nabla_{e_m} \nabla_{E_k} \nabla_{E_l} E_i - \nabla_{e_m} \nabla_{E_l} \nabla_{E_k} E_i - \nabla_{e_m} \nabla_{E_{kl}} E_i, e_j \rangle$$

(for the definition of the covariant derivative of a tensor see Exercise 9.3).

(b) Using (a) and the Riemannian curvature tensor, derive

$$\nabla R(e_i, e_j, e_k, e_l, e_m) + \nabla R(e_i, e_j, e_l, e_m, e_k) + \nabla R(e_i, e_j, e_m, e_k, e_l) = \langle \nabla_{[E_{mk}, E_l] + [E_{kl}, E_m] + [E_{lm}, E_k]} E_i, e_j \rangle$$

(c) Use Jacobi identity and linearity to prove the *Second Bianchi Identity*:

$$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W) = 0,$$

for $X, Y, Z, W, T$ vector fields on $M$.

**8.3. Schur Theorem**

Let $(M, g)$ be a connected Riemannian manifold of dimension $n \geq 3$ with the following property: there is a function $f : M \to \mathbb{R}$ such that, for every $p \in M$, the sectional curvature of **all** 2-planes $\Pi \subset T_pM$ satisfies

$$K(\Sigma) = f(p).$$

(a) Define $R(X, Y, Z, W) = \langle R(X, Y)Z, W \rangle$ and

$$R'(X, Y, Z, W) = \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle.$$

Use Exercises 4.3 and 7.4 to show that $\nabla R(X, Y, Z, W, U) = (Uf)R'(X, Y, Z, W)$.

(b) Use the Second Bianchi Identity (see Exercise 8.2) to show that

$$(Tf)(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle) + (Zf)(\langle X, T \rangle \langle Y, W \rangle - \langle X, W \rangle \langle Y, T \rangle) + (Wf)(\langle X, Z \rangle \langle Y, T \rangle - \langle X, T \rangle \langle Y, Z \rangle) = 0.$$

(c) Fix a point $p \in M$ and choose $X(p), Z(p) \in T_PM$ arbitrary. Since $n \geq 3$, we can choose $W, Y$ such that

$$\langle Z(p), W(p) \rangle_p = \langle Z(p), Y(p) \rangle_p = \langle Y(p), W(p) \rangle_p = 0,$$

and $\|Y(p)\| = 1$. Choose $T = Y$. Show that this choice yields

$$\langle (Wf)(p)Z(p) - (Zf)(p)W(p), X(p) \rangle (p) = 0,$$

and conclude that we have $(Zf)(p) = 0$.

(d) Prove *Schur Theorem*: show that $f$ is a constant function, i.e., there is a $C \in \mathbb{R}$ such that $f(p) = C$ for all $p \in M$.