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Riemannian Geometry IV, Term 1 (Section 4)

4 Levi-Civita connection and parallel transport

4.1 Levi-Civita connection

Example 4.1. Given a vector field $X = \sum a_i(p)\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathbb{R}^n)$ and a vector $v \in T_p\mathbb{R}^n$ define the covariant derivative of X in direction v in $\mathbb{R}^n$ by $\nabla_v(X) = \lim\limits_{t\to 0}\frac{X(p+tv)-X(p)}{t} = \sum v(a_i)\frac{\partial}{\partial x_i}\big|_p \in T_p\mathbb{R}^n$.

Proposition 4.2. *The covariant derivative $\nabla_v X$ in $\mathbb{R}^n$ satisfies all the properties (a)–(e) listed below in Definition 4.3 and Theorem 4.4.*

Definition 4.3. Let M be a smooth manifold. A map $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$, $(X,Y)\mapsto \nabla_X Y$ is affine connection if for all $X,Y,Z \in \mathfrak{X}(M)$ and $f,g\in C^\infty(M)$ holds

(a) $\nabla_X(Y+Z) = \nabla_X(Y) + \nabla_X(Z)$

(b) $\nabla_X(fY) = X(f)Y(p) + f(p)\nabla_X Y$

(c) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$

Theorem 4.4 (Levi-Civita, Fundamental Theorem of Riemannian Geometry)**.** *Let (M,g) be a Riemannian manifold. There exists a unique affine connection ∇ on M with the additional properties for all $X,Y,Z\in\mathfrak{X}(M)$:*

(d) $Z(\langle X,Y\rangle) = \langle\nabla_Z X, Y\rangle + \langle X,\nabla_Z Y\rangle$ *(Riemannian property);*

(e) $\nabla_X Y - \nabla_Y X = [X,Y]$ *(∇ is torsion-free).*

This connection is called Levi-Civita connection of (M,g).

Remark 4.5. Properties of Levi-Civita connection in $\mathbb{R}^n$ and in $M\subset\mathbb{R}^n$ with induced metric.

4.2 Christoffel symbols

Definition 4.6. Let ∇ be the Levi-Civita connection on (M,g), and let $\varphi : U \to V$ be a coordinate chart with coordinates $\varphi = (x_1,\dots,x_n)$. Since $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}(p) \in T_pM$, there exists a uniquely determined collection of functions $\Gamma_{ij}^k \in C^\infty(U)$ s.t. $\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}(p) = \sum_{k=1}^n \Gamma_{ij}^k(p)\frac{\partial}{\partial x_k}(p)$. These functions are called Christoffel symbols of ∇ with respect to the chart φ.

Remark. Christoffel symbols determine ∇ since $\nabla_{\sum\limits_{i=1}^n a_i\frac{\partial}{\partial x_i}}\sum\limits_{j=1}^n b_j\frac{\partial}{\partial x_j} = \sum\limits_{i,j} a_i\frac{\partial b_j}{\partial x_i}\frac{\partial}{\partial x_j} + \sum\limits_{i,j,k} a_i b_j\Gamma_{ij}^k\frac{\partial}{\partial x_k}$.

Proposition 4.7.

$$\Gamma_{ij}^k = \frac{1}{2}\sum_m g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m}),$$

where $g_{ab,c} = \frac{\partial}{\partial x_c}g_{ab}$ and $(g^{ij}) = (g_{ij})^{-1}$, i.e. $\{g^{ij}\}$ are the elements of the matrix inverse to (g_{ij}).

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Example 4.8. In $\mathbb{R}^n$, $\Gamma_{ij}^k \equiv 0$ for all i,j,k. Computation of Γ_{ij}^k in $S^2\subset\mathbb{R}^3$ with induced metric.

4.3 Parallel transport

Definition 4.9. Let $c : (a, b) \to M$ be a smooth curve. A smooth map $X : (a, b) \to TM$ with $X(t) \in T_{c(t)}M$ is called a vector field along c. These fields form a vector space $\mathfrak{X}_c(M)$.

Example 4.10. $c'(t) \in \mathfrak{X}_c(M)$.

Proposition 4.11. *Let (M, g) be a Riemannian manifold, let ∇ be the Levi-Civita connection, $c : (a, b) \to M$ be a smooth curve. There exists a unique map $\frac{D}{dt} : \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ satisfying*

(a) $\frac{D}{dt}(\alpha X + Y) = \alpha \frac{D}{dt} X + \frac{D}{dt} Y$ *for any* $\alpha \in \mathbb{R}$.

(b) $\frac{D}{dt}(fX) = f'(t)X + f \frac{D}{dt} X$ *for every* $f \in C^\infty(M)$.

(c) If $\widetilde{X} \in \mathfrak{X}(M)$ *is a local extension of* X
(i.e. there exists $t_0 \in (a, b)$ *and* $\varepsilon > 0$ *such that* $X(t) = \widetilde{X}\big|_{c(t)}$ *for all* $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$*)*
then $(\frac{D}{dt} X)(t_0) = \nabla_{c'(t_0)} \widetilde{X}$.

This map $\frac{D}{dt} : \mathfrak{X}_c(M) \to \mathfrak{X}_c(M)$ is called the covariant derivative along the curve c.

Example 4.12. Covariant derivative in $\mathbb{R}^n$.

Definition 4.13. Let $X \in \mathfrak{X}_c(M)$. If $\frac{D}{dt} X = 0$ then X is said to be parallel along c.

Example 4.14. A vector field X in $\mathbb{R}^n$ is parallel along a curve if and only if X is constant.

Theorem 4.15. *Let $c : [a, b] \to M$ be a smooth curve, $v \in T_{c(a)}M$. There exists a unique vector field $X \in \mathfrak{X}_c(M)$ parallel along c with $X(a) = v$.*

Corollary 4.16. *Parallel vector fields form a vector space of dimension n (where n is the dimension of (M, g)).*

Definition 4.17. Let $c : [a, b] \to M$ be a smooth curve. A linear map $P_c : T_{c(a)}M \to T_{c(b)}M$ defined by $P_c(v) = X(b)$, where $X \in \mathfrak{X}_c(M)$ is parallel along c with $X(a) = v$, is called a parallel transport along c.

Remark. The parallel transport P_c depends on the curve c (not only on its endpoints).

Proposition 4.18. *The parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$ is a linear isometry between $T_{c(a)}M$ and $T_{c(b)}M$, i.e. $g_{c(a)}(v, w) = g_{c(b)}(P_c v, P_c w)$.*