

## Riemannian Geometry IV, Term 1 (Section 5)

### 5 Geodesics

#### 5.1 Geodesics as solutions of ODE's

**Definition 5.1.** Given  $(M, g)$ , a curve  $c : [a, b] \rightarrow M$  is a geodesic if  $\frac{D}{dt}c'(t) = 0$  for all  $t \in [a, b]$  (i.e.,  $c'(t) \in \mathfrak{X}_c(M)$  is parallel along  $c$ ).

**Lemma 5.2.** *If  $c$  is a geodesic then  $c$  is parametrized proportionally to the arc length.*

**Theorem 5.3.** *Given a Riemannian manifold  $(M, g)$ ,  $p \in M$ ,  $v \in T_pM$ , there exists  $\varepsilon > 0$  and a unique geodesic  $c : (-\varepsilon, \varepsilon) \rightarrow M$  such that  $c(0) = p$ ,  $c'(0) = v$ .*

**Examples 5.4–5.5** Geodesics in Euclidean space, on a sphere, and in the upper half-plane model  $\mathbb{H}^2$ .

#### 5.2 Geodesics as distance-minimizing curves. First variation formula of the length

**Definition 5.6.** Let  $c : [a, b] \rightarrow M$  be a smooth curve. A smooth map  $F : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$  is a (smooth) variation of  $c$  if  $F(0, t) = c(t)$ . Variation is proper if  $F(s, a) = c(a)$  and  $F(s, b) = c(b)$  for all  $s \in (-\varepsilon, \varepsilon)$ .

Variation can be considered as a family of the curves  $F_s(t) = F(s, t)$ .

**Definition 5.7.** A variational vector field  $X \in \mathfrak{X}_c(M)$  of a variation  $F$  is defined by  $X(t) = \frac{\partial F}{\partial s}(0, t)$ .

**Definition 5.8.** The length  $l : (-\varepsilon, \varepsilon) \rightarrow [0, \infty)$  and energy  $E : (-\varepsilon, \varepsilon) \rightarrow [0, \infty)$  of a variation  $F$  are defined by

$$l(s) = \int_a^b \left\| \frac{\partial F}{\partial t}(s, t) \right\| dt, \quad E(s) = \frac{1}{2} \int_a^b \left\| \frac{\partial F}{\partial t}(s, t) \right\|^2 dt$$

**Remark.**  $l(s)$  is the length of the curve  $F_s(t)$ .

**Theorem 5.9.** *A smooth curve  $c$  is a geodesic if and only if  $c$  is parametrized proportionally to the arc length and  $l'(0) = 0$  for every proper variation of  $c$ .*

**Corollary 5.10.** *Let  $c : [a, b] \rightarrow M$  be the shortest curve from  $c(a)$  to  $c(b)$ , and  $c$  is parametrized proportionally to the arc length. Then  $c$  is geodesic.*

**Remark.** The converse is false (e.g., on the sphere).

**Lemma 5.11** (Symmetry Lemma). *Let  $W \subset \mathbb{R}^2$  be an open set and  $F : W \rightarrow M$ ,  $(s, t) \mapsto F(s, t)$ , be a smooth map. Let  $\frac{D}{dt}$  be the covariant derivative along  $F_s(t)$  and  $\frac{D}{ds}$  be the covariant derivative along  $F_t(s)$ . Then  $\frac{D}{dt} \frac{\partial F}{\partial s} = \frac{D}{ds} \frac{\partial F}{\partial t}$ .*

**Theorem 5.12** (First variation formula of length). *Let  $F : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$  be a variation of a smooth curve  $c(t)$ ,  $c'(t) \neq 0$ . Let  $X(t)$  be its variational vector field and  $l : (-\varepsilon, \varepsilon) \rightarrow [0, \infty)$  its length. Then*

$$l'(0) = \int_a^b \frac{1}{\|c'(t)\|} \frac{d}{dt} \langle X(t), c'(t) \rangle dt - \int_a^b \frac{1}{\|c'(t)\|} \langle X(t), \frac{D}{dt} c'(t) \rangle dt$$

**Corollary 5.13.** (a) *If  $c(t)$  is parametrized proportionally to the arc length,  $\|c'(t)\| \equiv c$ , then*

$$l'(0) = \frac{1}{c} \langle X(b), c'(b) \rangle - \frac{1}{c} \langle X(a), c'(a) \rangle - \frac{1}{c} \int_a^b \langle X(t), \frac{D}{dt} c'(t) \rangle dt;$$

(b) *if  $c(t)$  is geodesic, then  $l'(0) = \frac{1}{c} \langle X(b), c'(b) \rangle - \frac{1}{c} \langle X(a), c'(a) \rangle$ ;*

(c) *if  $F$  is proper and  $c$  is parametrized proportionally to the arc length, then  $l'(0) = -\frac{1}{c} \int_a^b \langle X(t), \frac{D}{dt} c'(t) \rangle dt$ ;*

(d) *if  $F$  is proper and  $c$  is geodesic, then  $l'(0) = 0$ .*

**Lemma 5.14.** *Any vector field  $X$  along  $c(t)$  with  $X(a) = X(b) = 0$  is a variational vector field for some proper variation  $F$ .*

### 5.3 Exponential map and Gauss Lemma

Let  $p \in M$ ,  $v \in T_p M$ . Denote by  $c_v(t)$  the unique maximal geodesic (i.e., the domain is maximal) with  $c_v(0) = p$ ,  $c'_v(0) = v$ .

**Definition 5.15.** If  $c_v(1)$  exists, define  $\exp_p : T_p M \rightarrow M$  by  $\exp_p(v) = c_v(1)$ , the exponential map at  $p$ .

**Example 5.16.** Exponential map on the sphere  $S^2$ : length of  $c_v$  from  $p$  to  $c_v(1)$  equals  $\|v\|$ .

**Notation.**  $B_r(0_p) = \{v \in T_p M \mid \|v\| < r\} \subset T_p M$  is a ball of radius  $r$  centered at  $0_p$ .

**Proposition 5.17.** (without proof)

*For any  $p \in (M, g)$  there exists  $r > 0$  such that  $\exp_p : B_r(0_p) \rightarrow \exp_p(B_r(0_p))$  is a diffeomorphism.*

**Example.** On  $S^2$  the set  $\exp_p(B_{\pi/2}(0_p))$  is a hemisphere, so that every geodesic starting from  $p$  is orthogonal to the boundary of this set.

**Theorem 5.18** (Gauss Lemma). *Let  $(M, g)$  be a Riemannian manifold,  $p \in M$ , and let  $\varepsilon > 0$  be such that  $\exp_p : B_\varepsilon(0_p) \rightarrow \exp_p(B_\varepsilon(0_p))$  is a diffeomorphism. Define  $A_\delta = \{\exp_p(w) \mid \|w\| = \delta\}$  for every  $0 < \delta < \varepsilon$ . Then every radial geodesic  $c : t \mapsto \exp_p(tv)$ ,  $t \geq 0$ , is orthogonal to  $A_\delta$ .*

**Remark 5.19.** The curve  $c_v(t) = \exp_p(tv)$  is indeed geodesic; every geodesic  $\gamma$  through  $p$  can be written as  $\gamma(t) = \exp_p(tv)$  for appropriate  $w \in T_p M$ .

**Definition.** Denote  $B_\varepsilon(p) = \exp_p(B_\varepsilon(0_p)) \subset M$ , a geodesic ball.

**Lemma 5.20.** *Let  $(M, g)$  be a Riemannian manifold and  $p \in M$ . Let  $\varepsilon > 0$  be small enough such that  $\exp_p : B_\varepsilon(0_p) \rightarrow B_\varepsilon(p) \subset M$  is a diffeomorphism. Let  $\gamma : [0, 1] \rightarrow B_\varepsilon(p) \setminus \{p\}$  be any curve. Then there exists a curve  $v : [0, 1] \rightarrow T_p M$ ,  $\|v(s)\| = 1$  for all  $s \in [0, 1]$ , and a positive function  $r : [0, 1] \rightarrow \mathbb{R}_+$ , such that  $\gamma(s) = \exp_p(r(s)v(s))$ .*

**Lemma 5.21.** Let  $r : [0, 1] \rightarrow \mathbb{R}_+$ ,  $v : [0, 1] \rightarrow S_p M = \{w \in T_p M \mid \|w\| = 1\}$ . Define  $\gamma : [0, 1] \rightarrow B_\varepsilon(p) \setminus \{p\}$  by  $\gamma(s) = \exp_p(r(s)v(s))$ . Then the length  $l(\gamma) \geq |r(1) - r(0)|$ , and the equality holds if and only if  $\gamma$  is a reparametrization of a radial geodesic (i.e.  $v(s) \equiv \|v(0)\|$  and  $r(s)$  is a strictly increasing or decreasing function).

**Corollary 5.22.** Given a point  $p \in M$ , there exists  $\varepsilon > 0$  such that for any  $q \in B_\varepsilon(p)$  there exists a curve  $c(t)$  connecting  $p$  and  $q$  and satisfying  $l(c) = d(p, q)$ . (This curve is a radial geodesic).

**Remark.** According to Corollary 5.22, there is  $\varepsilon > 0$  such that  $B_\varepsilon(p)$  coincides with  $\varepsilon$ -ball at  $p$ , i.e. with  $\{q \in M \mid d(p, q) < \varepsilon\}$ .

**Proposition 5.23.** (without proof)

Let  $p \in M$ . Then there is an open neighborhood  $U$  of  $p$  and  $\varepsilon > 0$  such that  $\forall q \in U \exp_q : B_\varepsilon(0_q) \rightarrow B_\varepsilon(q)$  is a diffeomorphism.

## 5.4 Hopf-Rinow Theorem

**Definition 5.24.** A geodesic  $c : [a, b] \rightarrow M$  is minimal if  $l(c) = d(c(a), c(b))$ . A geodesic  $c : \mathbb{R} \rightarrow M$  is minimal if its restriction  $c|_{[a, b]}$  is minimal for each segment  $[a, b] \subset \mathbb{R}$ .

**Example.** No minimal geodesics in  $S^2$ , all geodesics in  $\mathbb{E}^2$  are minimal.

**Definition 5.25.** A Riemannian manifold  $(M, g)$  is geodesically complete if every geodesic  $c : [a, b] \rightarrow M$  can be extended to a geodesic  $\tilde{c} : \mathbb{R} \rightarrow M$  (i.e. can be extended infinitely in both directions). Equivalently,  $\exp_p$  is defined on the whole  $T_p M$  for all  $p \in M$ .

**Theorem 5.26 (Hopf-Rinow).** Let  $(M, g)$  be a connected Riemannian manifold with distance function  $d$ . Then the following are equivalent:

- (a)  $(M, g)$  is complete (i.e. every Cauchy sequence converges);
- (b) every closed and bounded subset is compact;
- (c)  $(M, g)$  is geodesically complete.

Moreover, every of the conditions above implies

- (d) for every  $p, q \in M$  there exists a minimal geodesic connecting  $p$  and  $q$ .

**Remark.** A geodesic in (d) may not be unique. Further, (d) does not imply (c).

**Remark.** Theorem 5.26 uses the following notions defined in a metric space:

- $\{x_i\}$ ,  $x_i \in M$ , is a Cauchy sequence if  $\forall \varepsilon > 0 \exists N \forall m, n > N \quad d(x_m, x_n) < \varepsilon$ ;
- a set  $A \subset M$  is bounded if  $A \subset B_r(p)$  for some  $r > 0$ ,  $p \in M$ ;
- a set  $A \subset M$  is closed if  $\{x_n \in A, x_n \rightarrow x\} \Rightarrow x \in A$ ;
- a set  $A \subset M$  is compact if each open cover has a finite subcover;

**Exercise.** A compact metric space is complete.