Durham University
Pavel Tumarkin

Epiphany 2020

Riemannian Geometry IV, Term 2 (Section 10, non-examinable)

10 Curvature and geometry

10.1 Index form

Definition 10.1. Recall (see the proof of Second Variational Formula) that given a geodesic $c : [0,a] \to M$ there exists a symmetric bilinear form on $\mathfrak{X}_c(M)$ given by $I_a(V,W) = \int\limits_0^a (\langle \frac{D}{dt}V, \frac{D}{dt}W\rangle + \langle R(V,c')c', W\rangle)dt$.

The quadratic form $I_a(V,V)$ is called an index form.

Definition 10.2. The index of I_a is the maximal dimension of a subspace of $\mathfrak{X}_c(M)$ on which I_a is negative definite (i.e., negative inertia index).

Theorem 10.3 (Morse Index Theorem)**.** *The index of I_a is finite and equals the number of points $c(t)$, $0 < t < a$, conjugate to $c(0)$ counted with multiplicities.*

Corollary 10.4. *The set of conjugate points along a geodesic is discrete.*

Lemma 10.5 (Index Lemma)**.** *Let $c : [0,a] \to M$ be a geodesic containing no conjugate points to $c(0)$. Let $J \in J_c$ be an orthogonal Jacobi field. Let V be a piecewise smooth vector field on c, $\langle V, c'\rangle = 0$. Suppose also $J(0) = V(0) = 0$, $J(t_0) = V(t_0)$ for some $t_0 \in (0,a]$.*

Then $I_{t_0}(J,J) \le I_{t_0}(V,V)$, where equality holds only if $V = J$ on $[0,a]$.

10.2 Rauch Comparison Theorem

Theorem 10.6 (Rauch Comparison Theorem)**.** *Let $c : [0,a] \to M^n$ and $\tilde{c} : [0,a] \to \widetilde{M}^m$ be two unit speed geodesics, and let $J : [0,a] \to TM$ and $\widetilde{J} : [0,a] \to T\widetilde{M}$ be two orthogonal Jacobi fields along c and $\widetilde{c}$ with $J(0) = \widetilde{J}(0) = 0$, $\|\frac{D}{dt}J(0)\| = \|\frac{D}{dt}\widetilde{J}(0)\|$. Assume that $\widetilde{J}$ does not vanish on $(0,a)$, and that for any $t \in [0,a]$ the inequality $K_M(\Pi) \le K_{\widetilde{M}}(\widetilde{\Pi})$ holds for all 2-planes $\Pi \subset T_{c(t)}M$ and $\widetilde{\Pi} \subset T_{\tilde{c}(t)}\widetilde{M}$. Then $\|J(t)\| \ge \|\widetilde{J}(t)\|$ for all $t \in [0,a]$.*

Remark. In particular, Theorem 10.6 implies that if the geodesic $\widetilde{c}$ has no conjugate points between $\widetilde{c}(0)$ and $\widetilde{c}(a)$, then c has no conjugate points between $c(0)$ and $c(a)$ either.

Notation. In what follows, by K we will denote the sectional curvature on M as a real-valued function on the set of all 2-planes in TM. The notation $K \le K_0$ means that the function K is uniformly bounded from above, etc.

Corollary 10.7. *(a) Let M satisfy $0 < K_{\min} \le K \le K_{\max}$, $c : [0,a] \to M$ is a geodesic. Then for any two conjugate points along c the distance d between them satisfies*

$$\frac{\pi}{\sqrt{K_{\max}}} \le d \le \frac{\pi}{\sqrt{K_{\min}}}$$

(b) Let M be complete, simply-connected, and satisfy $K \le 0$. Then for any geodesic triangle in M the sum of angles does not exceed π (where the inequality is strict if $K < 0$.

Idea of proof of Corollary 10.7(a). Apply Theorem 10.6 to compare M to spheres of radius $1/\sqrt{K_{\max}}$ and $1/\sqrt{K_{\min}}$ and use the remark above.

□

Corollary 10.8 (Reformulation of Theorem 10.6)**.** *In the assumptions of Theorem 10.6, one has*

$$\|(D\exp_{c(0)} tc'(0))\, t\frac{D}{dt}J(0)\| \geq \|(D\exp_{\widetilde{c}(0)} t\widetilde{c}'(0))\, t\frac{D}{dt}\widetilde{J}(0)\|$$

Corollary 10.9. *Let M be complete, simply-connected, and satisfy $K \leq 0$. Let $p \in M$, $w \in T_pM$, $u \in T_pM = T_w(T_pM)$. Then*

$$\|(D\exp_p w)\, u\| \geq \|u\|$$

Proof. Apply Corollary 10.8 to $\widetilde{M} = T_pM$ with constant metric $\langle \cdot, \cdot \rangle_p$ (and thus zero curvature), $w = tc'(0)$, $u = t\frac{D}{dt}J(0)$. As the exponential map on a vector space $\widetilde{M} = T_pM = \mathbb{R}^n$ is an identity map, its differential is also an identity map, so the result follows.

□

Remark. Roughly speaking, Corollary 10.9 says that the differential of the exponential map in negative curvature increases magnitudes of tangent vectors. In particular, this implies that the exponential map itself increases lengths of curves, i.e. for any curve γ in T_pM one has $l(\gamma) \leq l(\exp_p(\gamma)$.

Idea of proof of Corollary 10.7(b). Let ABC be a triangle in M, $C = p$. Take a (Euclidean) triangle OPQ in T_pM, where $O = 0$, $\exp_p Q = A$, $\exp_p P = B$. Define a curve γ between P and Q by $\exp_p \gamma = AB$. Then $|PQ| \leq l(\gamma) \leq l(AB) = l(\exp_p \gamma)$ by Corollary 10.9.

Let $A'B'C'$ be a Euclidean triangle with $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|AC| = |A'C'|$. Applying now the (Euclidean) cosine law to OPQ, we see that $\angle C \leq \angle C'$. Similar inequalities hold for other two angles, so $\angle A + \angle B + \angle C \leq \angle A' + \angle B' + \angle C' = \pi$.

□

The following theorem generalizes Corollary 10.9 to non-negative curvature.

Theorem 10.10. *Let M and $\widetilde{M}$ be two complete Riemannian manifolds, where $\widetilde{M}$ has constant sectional curvature $\widetilde{K}$, and the curvature K on M satisfies $K \leq \widetilde{K}$. Let $p \in M$, $\widetilde{p} \in \widetilde{M}$.*

Let U be a ball in T_pM centered at 0_p s.t. $\exp_p$ is a diffeomorphism on U. Without loss of generality we can assume that $\exp_{\widetilde{p}}$ is also a diffeomorphism on the ball $\widetilde{U}$ of the same radius in $T_{\widetilde{p}}\widetilde{M}$, so we may identify $\widetilde{U}$ and U.

Let pqr be a geodesic triangle in $\exp_p(U) \subset M$ with sides being minimal geodesics. Let the geodesic $c : [a, b] \to M$ be the side rq with $c(a) = r$ and $c(b) = q$, and denote by $v(s)$ the curve in U such that $\exp_p v(s) = c(s)$. Define $\widetilde{c}(s) = \exp_{\widetilde{p}} v(s)$, and let $\widetilde{r} = \exp_{\widetilde{p}} v(a)$, $\widetilde{q} = \exp_{\widetilde{p}} v(b)$. Then $d_{\widetilde{M}}(\widetilde{r}, \widetilde{q}) \leq d_M(r, q)$.

Proof. Define $F(s,t) = \exp_p tv(s)$ and $\widetilde{F}(s,t) = \exp_{\widetilde{p}} tv(s)$. For a given s_0, we can consider $F(s,t)$ and $\widetilde{F}(s,t)$ as geodesic variations of $F(s_0,t)$ and $\widetilde{F}(s_0,t)$ respectively. Thus, their variational vector fields $J_{s_0} = \frac{\partial F}{\partial s}(s_0, t)$ and $\widetilde{J}_{s_0} = \frac{\partial \widetilde{F}}{\partial s}(s_0, t)$ are Jacobi fields along $F(s_0,t)$ and $\widetilde{F}(s_0,t)$ respectively.

Observe that J_{s_0} and $\widetilde{J}_{s_0}$ satisfy the assumptions of Rauch Comparison Theorem, and thus $\|J_{s_0}(t)\| \geq \|\widetilde{J}_{s_0}(t)\|$. In particular, $\|J_{s_0}(1)\| \geq \|\widetilde{J}_{s_0}(1)\|$.

Observe now that $J_{s_0}(1) = \frac{\partial \exp_p v}{\partial s}(s_0) = c'(s_0)$, and similarly $\widetilde{J}_{s_0}(1) = \widetilde{c}'(s_0)$, so the inequality above leads to $\|c'(s)\| \geq \|\widetilde{c}'(s)\|$ for any $s \in [a, b]$, which implies

$$d_{\widetilde{M}}(\widetilde{r}, \widetilde{q}) \leq l(\widetilde{c}) \leq l(c) = d_M(r, q)$$

□

10.3 Injectivity radius

Definition 10.11. Let M be a complete Riemannian manifold. The injectivity radius of a point $p \in M$ is $i_p = \sup\{r \geq 0 \mid \exp_p \text{ is diffeo in } B_r(0_p)\} = \inf\limits_{q \in C_m(p)} d(p,q)$, where $C_m(p)$ is the cut locus of p.

The injectivity radius of M is $i(M) = \inf\limits_p i_p = \inf\limits_{p \in M} d(p, C_m(p))$.

Remark. The notion of injectivity radius for non-complete M does not make too much sense: it is always equal to zero.

Example 10.12. $i(S^2) = \pi$; $i(\mathbb{R}^2) = i(\mathbb{H}^2) = \infty$; $i(\mathbb{T}^2) = 1/2$.

Proposition 10.13. *Let M be complete with sectional curvature K satisfying $0 < K_{\min} \leq K \leq K_{\max}$. Then at least one of the following holds:*

(a) $i(M) \geq \pi/\sqrt{K_{max}}$, or

(b) there exists a shortest closed geodesic $c \subset M$ s.t. $i(M) = \frac{1}{2} l(c)$.

Lemma 10.14 (Klingenberg, 1961)**.** *Let M be a compact simply-connected Riemannian manifold of dimension $n \geq 3$, and let $1/4 < K \leq 1$. Then $i(M) \geq \pi$.*

Remark. If n is even and M is orientable then it suffices for M to satisfy $0 < K \leq 1$.

10.4 Sphere Theorem

Theorem 10.15 (Berger, Klingenberg, 1961)**.** *Let M be a compact simply-connected Riemannian n-dimensional manifold with $\frac{1}{4} < K \leq 1$. Then M is homeomorphic to S^n.*

Remark. (a) In fact, a stronger result is valid: M is diffeomorphic to S^n (Brendle, Schoen, 2009).

(b) The Sphere Theorem does not hold in the assumptions $\frac{1}{4} \leq K(\Pi) \leq 1$ (see Example 10.18).

(c) The theorem obviously holds in the assumptions $\frac{\delta}{4} < K(\Pi) \leq \delta$ for any $\delta > 0$.

(d) In dimension $n = 2$ stronger result holds: if $K \geq 0$ for all $p \in M$ and $K > 0$ in at least one point, then M is homeomorphic to S^2.

The proof of the Sphere Theorem is based on the following two lemmas.

Lemma 10.16. *Let M be a compact Riemannian manifold, let $p, q \in M$ be such that* $\operatorname{diam} M = d(p,q)$*. Then for any $w \in T_pM$ there exists a minimal geodesic $c : [0, d(p,q)] \to M$, $c(0) = p$, $c(d(p,q)) = q$, such that $\langle w, c'(0) \rangle \geq 0$.*

Lemma 10.17. *Let M be a compact simply-connected Riemannian manifold with sectional curvature satisfying $\frac{1}{4} < \delta \leq K \leq 1$, let $p, q \in M$ be such that* $\operatorname{diam} M = d(p,q)$*. Choose any $\rho \in (\pi/2\sqrt{\delta}, \pi)$. Then $M = B_\rho(p) \cup B_\rho(q)$.*

In other words, Lemma 10.17 says that M is covered by two ρ-balls centered at any two "opposite" points of M.

Sketch of a proof of Lemma 10.17. By Lemma 10.14, the injectivity radius $i(M) \geq \pi$ (please note: this is the place the assumption $\delta > 1/4$ shows up), so since $< \pi$ both $B_\rho(p)$ and $B_\rho(q)$ are diffeomorphic to Euclidean balls. We need to show that these balls cover M, i.e. any $x \in M$ lies in at least one of these two balls. We will prove this by contradiction.

Take a minimal geodesic between p and q, let q'' be its intersection with a sphere $\partial B_\rho(q)$. Observe that, since $K \geq \delta$, Ricci curvature on M is bounded below by $(n-1)\delta$, and thus, by Bonnet – Myers Theorem, diam $M \leq \pi/\sqrt{\delta} \leq 2\rho$. In particular, this implies that $q'' \in B_\rho(p)$.

Now suppose that the lemma fails, i.e. there exists $x \in M$ such that $d(x,p) \geq \rho$ and $d(x,q) \geq \rho$ (we may assume $d(x,p) \geq d(x,q)$ without loss of generality). Take a minimal geodesic from x to q, denote its intersection with $\partial B_\rho(q)$ by q'. If we assume that $q' \in B_\rho(p)$, then $d(x,q') > d(x, B_\rho(p))$ as q' is not the closest point of $B_\rho(p)$ to x (since $B_\rho(p)$ is open). In view of our assumption $d(x,p) \geq d(x,q)$, this implies

$$d(x,q') > d(x, B_\rho(p)) \geq d(x, B_\rho(q)) = d(x,q'),$$

where the last equality uses Gauss Lemma. The contradiction shows that $q' \notin B_\rho(p)$.

Therefore, in the assumption that the lemma fails, we have found points q', q'' on the sphere $\partial B_\rho(q)$, such that $d(q'',p) < \rho$ and $d(q',p) \geq \rho$. By (path-)connectedness of the sphere and continuity of the function "distance to a given point", there exists $x_0 \in \partial B_\rho(q)$ such that $d(x_0,p) = \rho$, i.e. $x_0 \in \partial B_\rho(q) \cap \partial B_\rho(p)$. We now forget about q', q'' and x, and show that the existence of x_0 leads to a contradiction.

Let c be a minimal geodesic between p and x_0. By Lemma 10.16, there exists a minimal geodesic γ from p to q such that $\langle \gamma'(0), c'(0) \rangle \geq 0$. Denote by y the intersection of γ with $\partial B_\rho(p)$. Since $K \geq \delta$, we can compare M to an n-sphere of curvature δ, obtaining $d(x_0, y) \leq \pi/2\sqrt{\delta} < \rho$ (here we use the fact the angle between c and γ is at mist $\pi/2$, and an analog of Theorem 10.10 for $K > \widetilde{K}$ – it also holds!).

Take a minimal shortest geodesic γ_0 connecting x_0 to a point of γ, denote by y_0 its intersection with γ. Clearly, $d(x_0,y_0) \leq d(x_0,y)$, which implies $d(x_0,y_0) < \rho$ due to the inequality above. Observe that the angle at y_0 between the two geodesics γ and γ_0 is equal to $\pi/2$, otherwise we could find a shorter path between x_0 and γ. Also, since $y_0 \in \gamma$ and the length of γ is less than 2ρ, y_0 must lie in at least one of the balls $B_\rho(p)$ and $B_\rho(q)$. Without loss of generality, assume that $y_0 \in B_\rho(p)$.

We can now consider a right-angled triangle with vertices p, x_0, y_0 formed by geodesics c, γ, γ_0. We also know that its sides px_0 and x_0y_0 are strictly shorter than ρ. Comparing this triangle with one on the sphere of curvature δ (and applying an analog of Theorem 10.10 again), we see that $d(x_0,p) < \rho$, which contradicts the definition of x_0.

□

Proof of the Sphere Theorem. The proof is now straightforward and is based on the following fact from topology: if a compact manifold is covered by two topological discs, then it is homeomorphic to a sphere. In view of Lemma 10.17, this completes the proof.

□

Example 10.18. Consider complex projective space $\mathbb{CP}^2 = \{(z_0 : z_1 : z_2) \mid z_i \in \mathbb{C}, (z_0, z_1, z_2) \neq 0\}$. This is a 2-dimensional complex manifold, which can be considered as a 4-dimensional real manifold. It can be assigned with *Fubini – Study metric*, which in the chart $z_0 \neq 0$ (or, equivalently, $z_0 = 1$) in complex coordinates is given by the following Hermitian matrix:

$$\tilde{G} = (\tilde{g}_{ij}) = \frac{1}{(1+|z_1|^2+|z_2|^2)^2} \begin{pmatrix} 1+|z_2|^2 & -\bar{z}_1 z_2 \\ -z_1\bar{z}_2 & 1+|z_1^2 \end{pmatrix}$$

(i.e. for tangent vectors v and w one has $\langle v, w \rangle = \bar{v}^t \tilde{G} w$).

Remark (Realification)**.** Every complex vector n-space V can be considered as real $2n$-space $V_\mathbb{R}$: if V has basis $\{v_1, \ldots, v_n\}$, then the collection of vectors $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ is a basis of $V_\mathbb{R}$ (this procedure is called *realification*). Every linear operator on V also acts on $V_\mathbb{R}$. More precisely, if A is a matrix of linear operator on V in the basis $\{v_1, \ldots, v_n\}$, then the matrix $A_\mathbb{R}$ of the corresponding linear operator on $V_{\mathbb{R}}$ in the basis $\{v_1, \dots, v_n, iv_1, \dots, iv_n\}$ is a block matrix $A_{\mathbb{R}} = \begin{pmatrix} \operatorname{Re} A & -\operatorname{Im} A \\ \operatorname{Im} A & \operatorname{Re} A \end{pmatrix}$.

For example, a rotation by an angle φ on a plane can be written as a multiplication by $e^{i\varphi}$ in $\mathbb{C}$ or as a matrix $\begin{pmatrix} \operatorname{Re} e^{i\varphi} & -\operatorname{Im} e^{i\varphi} \\ \operatorname{Im} e^{i\varphi} & \operatorname{Re} e^{i\varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ in $\mathbb{R}^2$.

Applying the described above procedure of realification to the matrix $\tilde{G}$, we obtain Fubini – Study metric in real coordinates (in the basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\}$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$):

$$G = (g_{ij}) = \frac{1}{(1 + x_1^2 + y_1^2 + x_2^2 + y_2^2)^2} \begin{pmatrix} 1 + x_2^2 + y_2^2 & -x_1x_2 - y_1y_2 & 0 & x_1y_2 - x_2y_1 \\ -x_1x_2 - y_1y_2 & 1 + x_1^2 + y_1^2 & -x_1y_2 + x_2y_1 & 0 \\ 0 & -x_1y_2 + x_2y_1 & 1 + x_2^2 + y_2^2 & -x_1x_2 - y_1y_2 \\ x_1y_2 - x_2y_1 & 0 & -x_1x_2 - y_1y_2 & 1 + x_1^2 + y_1^2 \end{pmatrix}.$$

Restricting G to the 2-plane spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial y_1}$ (*complex line*) we obtain four times the standard metric on the unit sphere, so $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}) = 4$. Restricting G to a *totally real plane* one can compute that the sectional curvature $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}) = 1$. Sectional curvature in any other direction is in between 4 and 1.

In fact, $\mathbb{CP}^2$ is not homeomorphic to a sphere, so this example shows that the Sphere Theorem does not hold in the assumptions $K_{\max}/K_{\min} = 4$.

10.5 Spaces of constant curvature

Theorem 10.19. *Let M be a complete simply-connected Riemannian manifold of <u>constant</u> sectional curvature K. Then*

1) if $K > 0$ then M is isometric to S^n (assuming $K = 1$);
2) if $K = 0$ then M is isometric to $\mathbb{E}^n$;
3) if $K < 0$ then M is isometric to $\mathbb{H}^n$ (assuming $K = -1$).

10.6 Comparison triangles

Definition 10.20. A <u>triangle</u> in a Riemannian manifold is a collection of 3 points with <u>minimal</u> geodesics connecting them. A <u>generalized triangle</u> is a collection of 3 points with <u>any</u> geodesics connecting them and satisfying triangle inequality.

Definition 10.21. A <u>comparison triangle</u> $p'q'r'$ for a generalized triangle $pqr \in M$ is a triangle in a space of constant curvature with sides of the same lengths.

Theorem 10.22 (Alexandrov, Toponogov, 1959)**.** *Let $K(\Pi) \geq 0$ for all $\Pi \in T_pM$ for all $p \in M$. Let $p_0, p_1, p_2 \in M$. Let p_3 lie between p_1 and p_2 (i.e. $d(p_1, p_3) + d(p_2, p_3) = d(p_1, p_2)$). Let p_0', p_1', p_2' be a comparison triangle in $\mathbb{E}^2$. Define p_3' by $d(p_i, p_3)_M = d(p_i', p_3')_{\mathbb{E}^2}$ for $i = 1, 2$. Then $d(p_0, p_3)_M \geq d(p_0', p_3')_{\mathbb{E}^2}$ (Alexandrov – Toponogov inequality). Conversely, if Alexandrov – Toponogov inequality holds for all p_0, p_1, p_2, p_3 then $K \geq 0$.*

Remark. (a) Dual statement for $K \leq 0$ with inverse AT-inequality.

(b) Equivalent conditions:

- smaller K implies smaller <u>angles</u>;
- smaller K implies bigger <u>circumference</u> of a circle of radius r;
- smaller K implies bigger <u>volume</u> of a ball or radius r.