Durham University
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Riemannian Geometry IV, Term 2 (Section 6)

6 Crash course: Basics about Lie groups

6.1 Left-invariant vector fields and Lie algebra

Definition 6.1. A Lie group G is a smooth manifold with a smooth group structure, i.e. the maps $G \times G \to G$, $(g, h) \mapsto gh$ and $G \to G$, $g \mapsto g^{-1}$ are smooth.

Examples. Matrix Lie groups $GL_n(\mathbb{R})$, $SL_n(\mathbb{R})$, $O_n(\mathbb{R})$, $SO_n(\mathbb{R})$.

Definition 6.2. Let G a be a Lie group, $g \in G$. Then the maps $L_g : G \to G$ and $R_g : G \to G$ defined by $L_g(h) = gh$ and $R_g(h) = hg$ are called left- and right-translation. L_g and R_g are diffeomorphisms of G.

Remark. (a) $L_{g^{-1}} \circ L_g = id_G$, $\quad L_{g_1}R_{g_2}(h) = R_{g_2}L_{g_1}(h) = g_1hg_2$.

(b) The differential $DL_g : T_hG \to T_{gh}G$ gives a natural identification of tangent spaces.

(c) Every diffeomorphism $\varphi : G \to G$ induces an action of the differential $D\varphi$ on $\mathfrak{X}(G)$, $D\varphi : X \mapsto D\varphi X$ in the following way: $D\varphi X(h) = D\varphi(\varphi^{-1}(h))X(\varphi^{-1}(h))$, or, equivalently, $D\varphi X(\varphi(h)) = D\varphi(h)X(h)$. In particular, DL_g acts on $\mathfrak{X}(G)$ by $DL_g : X \mapsto DL_gX$, where

$$DL_gX(h) = DL_g(g^{-1}h)X(g^{-1}h).$$

Example 6.3. Let $G \subset GL_n(\mathbb{R})$ be a matrix group, $v \in T_eG$. Then $DL_g(e)v = gv$.

Definition 6.4. A vector field $X \in \mathfrak{X}(G)$ is called left-invariant if for any $g \in G$ $\;DL_gX = X$, i.e. $DL_g(h)X(h) = X(gh)$.

Example. Let $G \subset GL_n(\mathbb{R})$ be a matrix group, $X \in \mathfrak{X}(G)$ is left-invariant. Then $X(gh) = gX(h)$, and in particular $X(g) = gX(e)$.

Remark 6.5. (a) Left-invariant vector fields on G form a vector space over $\mathbb{R}$.

(b) Left-invariant vector field is determined by its value at e: $X(g) = DL_g(e)X(e)$.

(c) Hence, the space of left-invariant vector fields on G can be identified with T_eG.

Definition 6.6. The space of left-invariant vector fields on G is called the Lie algebra of G and denoted by $\mathfrak{g}$.

Lemma 6.7. *Let M, N be smooth manifolds, $X \in \mathfrak{X}(M)$, $f \in C^\infty(N)$, $p \in M$, and let $\varphi : M \to N$ be a smooth map. Then*

$$(d\varphi(p)X(p))f = X(p)(f \circ \varphi)$$

Proposition 6.8. *Let X be a Lie group with Lie algebra $\mathfrak{g}$. Then for any $X, Y \in \mathfrak{g}$ the Lie bracket $[X, Y] \in \mathfrak{g}$. Consequently, $\mathfrak{g}$ is indeed a Lie algebra (see Definition 2.22).*

6.2 Lie group exponential map and adjoint representation

Definition 6.9. Define $\mathrm{Exp}: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $\mathrm{Exp}\,(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

Properties. (a) The infinite sum converges for any matrix $A \in M_n(\mathbb{R})$, so $\mathrm{Exp}\,(A)$ is well-defined;

(b) $\mathrm{Exp}\,(0) = I$;

(c) if $AB = BA$ then $\mathrm{Exp}\,(A+B) = \mathrm{Exp}\,(A) \cdot \mathrm{Exp}\,(B)$; in particular, $\mathrm{Exp}\,(-A)\mathrm{Exp}\,(A) = I$, so $\mathrm{Exp}\,(A) \in GL_n(\mathbb{R})$ for any $A \in M_n(\mathbb{R})$;

(d) note that $M_n(\mathbb{R}) = T_e GL_n(\mathbb{R})$, and from (c) we know that $\mathrm{Exp}\,(M_n(\mathbb{R})) \subset GL_n(\mathbb{R})$. One can check that this property holds for any matrix Lie group: if $G \subset GL_n(\mathbb{R})$, then $\mathrm{Exp}\,(T_e G) \subset G$.

Example 6.10. Computation of the exponent for a diagonalizable matrix.

Proposition 6.11. *Let G be a matrix Lie group. Let $v \in T_e G$ and let X be the unique left-invariant vector field on G with $X(e) = v$. Then the curve $c(t) = \mathrm{Exp}\,(tv) \in G$ satisfies $c(0) = e$, $c'(0) = v$ and $c'(t) = X(c(t))$.*

A curve of the form $c(t) = \mathrm{Exp}\,(tv)$ is called a 1-parameter subgroup of G with $c'(0) = v$.

Remark. For an abstract Lie group the exponential map can be defined as follows. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Let $v \in T_e G$ and let $X \in \mathfrak{g}$ be the unique left-invariant vector field with $X(e) = v$. Then there exists a unique curve $c_v : \mathbb{R} \to G$ with $c_v(0) = e$, $c_v'(t) = X(c_v(t))$ [without proof]. The curve c_v is called an integral curve of X. We define the exponential map by $\mathrm{Exp}\,(v) = c_v(1)$.

Definition 6.12. Let G be a Lie group. For $g \in G$ the adjoint representation $\mathrm{Ad}_g : T_e G \to T_e G$ is defined by

$$\mathrm{Ad}_g(w) = \frac{d}{dt}\Big|_{t=0} L_g R_{g^{-1}}(\mathrm{Exp}\,(tw)) = \frac{d}{dt}\Big|_{t=0} g\mathrm{Exp}\,(tw)g^{-1}.$$

For $v \in T_e G$ the adjoint representation $\mathrm{ad}_v : T_e G \to T_e G$ is defined by

$$\mathrm{ad}_v(w) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\mathrm{Exp}\,(tv)}(w) = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \mathrm{Exp}\,(tv)\mathrm{Exp}\,(sw)\mathrm{Exp}\,(-tv).$$

Example. Let $G \subset GL_n(\mathbb{R})$, $g \in G$, $v, w \in \mathbb{T}_e G$. Then $\mathrm{Ad}_g w = gwg^{-1}$, and $\mathrm{ad}_v w = [v, w]$.

Theorem 6.13 (without proof)**.** *Let G be a Lie group with a Lie algebra $\mathfrak{g}$. Then for all $X, Y \in \mathfrak{g}$ holds $\mathrm{ad}_{X(e)} Y(e) = [X, Y](e) \in T_e G$, i.e. by canonical identification of $\mathfrak{g}$ with $T_e G$ we have $\mathrm{ad}_X Y = [X, Y]$.*

Corollary. $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a representation of Lie algebra $\mathfrak{g}$.

Example 6.14. Theorem 6.13 for the case of a matrix Lie group.

6.3 Riemannian metrics on Lie groups

Definition 6.15. For a given inner product $\langle \cdot, \cdot \rangle_e$ on $T_e G$, define the inner product at $g \in G$ for $v, w \in T_g G$ by $\langle v, w \rangle_g = \langle DL_{g^{-1}}(g)v, DL_{g^{-1}}(g)w \rangle_e$. The family $(\langle \cdot, \cdot \rangle_g)_{g \in G}$ of inner products defines a left-invariant Riemannian metric on G.

Example. For $G \subset GL_n(\mathbb{R})$, one can define $\langle v, w \rangle_e = \mathrm{tr}\; v^t w$ for $v, w \in T_e G$, and thus a left-invariant metric $\langle x, y \rangle_g = \mathrm{tr}\,(g^{-1}x)^t g^{-1} y = \mathrm{tr}\; x^t (g^{-1})^t g^{-1} y$ for $x, y \in T_g G$. In particular, for $G = SO_n(\mathbb{R})$ this metric is constant: $\langle x, y \rangle_g = \mathrm{tr}\; x^t y$.

Remark 6.16. Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with a left-invariant metric. Then

(a) the diffeomorphisms $L_g : G \to G$ are isometries;

(b) for any two left-invariant vector fields $X, Y \in \mathfrak{g}$ the function $g \mapsto \langle X(g), Y(g) \rangle_g$ is constant.

Theorem 6.17 (without proof)**.** *Let G be a* <u>*compact*</u> *Lie group. Then G admits a bi-invariant Riemannian metric $\langle \cdot, \cdot \rangle_g$, i.e. both families of diffeomorphisms L_g and R_g are isometries.*

Corollary 6.18. *Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with bi-invariant metric, let $X, Y, Z \in \mathfrak{g}$. Then $\langle [X, Y], Z \rangle = -\langle [X, Z], Y \rangle$.*

Corollary 6.19. *Let $(G, \langle \cdot, \cdot \rangle)$ be a Lie group with bi-invariant metric and let ∇ be the Levi-Civita connection. Then for $X, Y \in \mathfrak{g}$ holds $\nabla_X Y = \frac{1}{2}[X, Y]$.*

Corollary 6.20. *(a) 1-parameter subgroups of G are exactly the geodesics of the bi-invariant metric on G passing through e;*

(b) the Lie group exponential map Exp *coincides with the Riemannian exponential map* $\exp_e$ *at the neutral element.*

6.4 Homogeneous spaces

Definition 6.21. A connected Riemannian manifold (M, g) is called <u>homogeneous</u> if the group $\mathrm{Isom}\,(M, g)$ of isometries of M acts transitively on M, i.e. for any $p, q \in M$ there exists $\varphi \in \mathrm{Isom}\,(M, g)$ s.t. $\varphi(p) = q$.

Examples. $\mathbb{E}^n$, S^n, Lie groups.

Given a Lie group G and a <u>closed</u> subgroup $H \subset G$, consider the set $M = G/H = \{gH \mid g \in G\}$. Then M is a smooth manifold (non-trivial theorem, uses that H is closed).

Theorem 6.22 (without proof)**.** *Left-invariant metrics on G/H are in one-to-one correspondence with* $\mathrm{Ad}\,(H)$*-invariant inner products on T_eG.*

Example 6.23. Let $G = SO_3(\mathbb{R})$, $H = SO_2(\mathbb{R}) \subset G$. Then $M = SO_3/SO_2 \approx S^2$. In general, $O_n(\mathbb{R})/O_{n-1}(\mathbb{R}) \approx S^{n-1}$.