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Riemannian Geometry IV, Term 2 (Sections 7-8)

7 Curvature

7.1 Riemann curvature tensor

Definition 7.1. Let (M, g) be a Riemannian manifold, let $\mathfrak{X}(M)$ be the space of vector fields on M, and let ∇ be the Levi-Civita connection. Define a map (Riemann curvature tensor) $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$.

Remark. R is linear in all variables, so, it is a tensor; moreover, $R(fX, gY)hZ = fghR(X, Y)Z$ for any $f, g, h \in C^\infty(M)$.

Lemma 7.2. *R has the following symmetries:*
(a) $R(X, Y)Z = -R(Y, X)Z$
(b) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$
(first Bianchi Identity)
(c) $\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle$
(d) $\langle R(X, Y)Z, W\rangle = -\langle R(Z, W)X, Y\rangle$

Definition 7.3. Define components of Riemann curvature tensor $R_{ijkl} = \langle R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\rangle$, and R^l_{ijk} by $R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} = \sum_l R^l_{ijk}\frac{\partial}{\partial x_l}$.

Then $R_{ijkl} = \sum_m R^l_{ijk} g_{ml}$ and $R^l_{ijk} = \sum_m R_{ijkm} g^{ml}$.

Example 7.4. Computation of components R_{ijks} and R^l_{ijk} for hyperbolic plane (in the upper half-plane model).

7.2 Sectional curvature

Definition 7.5. Let (M, g) be a Riemannian manifold, $p \in M$, $v_1, v_2 \in T_pM$, and let $\Pi \subset T_pM$ be the 2-plane spanned by v_1, v_2.
The sectional curvature of Π at p is $K(\Pi) = K(v_1, v_2) = \frac{\langle R(v_1,v_2)v_2,v_1\rangle}{\|v_1\|^2\|v_2\|^2-\langle v_1,v_2\rangle^2}$.

Proposition 7.6. *$K(\Pi)$ does not depend on the basis $\{v_1, v_2\}$ of Π.*

Examples. Sectional curvature of a 2-sphere and hyperbolic plane.

7.3 Ricci curvature

Given $v, w \in T_pM$ define a linear map $R(\cdot, v)w : T_pM \to T_pM$ by $u \mapsto R(u, v)w$.

Definition 7.7. Ricci curvature tensor $Ric(v, w)$ is the trace of the map $R(\cdot, v)w$: $Ric_p(v, w) = tr(R(\cdot, v)w)$.

In an orthonormal basis $\{u_i\}$, $Ric_p(v, w) = \sum_{j=1}^n \langle R(u_j, v)w, u_j\rangle$.

Definition 7.8. Ricci curvature at p is $Ric_p(v) = Ric_p(v, v) = \sum_{j=1}^n \langle R(u_j, v)w, u_j\rangle$

In an orthonormal basis $\{v = u_1, \dots, u_n\}$ we have $Ric_p(v) = \sum_{j=2}^n K(v, u_j)$.

Lemma 7.9. *$Ric(v, u)$ is a symmetric bilinear form (i.e. $Ric(v)$ is a quadratic form).*

Example. If $K(v, w)$ is constant ($= K$) and $\|v\| = 1$, then $Ric(v) = (n - 1)K$.

8 Bonnet – Myers Theorem

Theorem 8.1 (Bonnet – Myers, 1935)**.** *Let (M, g) be a connected, complete Riemannian manifold of dimension n.*
Suppose that $Ric(v) \geq \frac{n-1}{r^2}$ for all $v \in SM = \{w \in TM \mid \|w\| = 1\}$. Then diam M $(= \sup\limits_{p,q \in M} d(p,q)) \leq \pi r$.
In particular, M is bounded, so, it is compact (as it is complete).

Theorem 8.2 (Second variation formula of length)**.** *Let $c : [a, b] \to M$ be a geodesic parametrized by arc length, let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a proper variation of c, let $X(t) = \frac{\partial F}{\partial s}(0, t)$ be the variational vector field. Define $X^{\perp}(t) = X(t) - \langle X(t), c'(t)\rangle c'(t)$, the orthogonal component of $X(t)$. Let $l(s)$ be the length of the variation.*

Then $l''(0) = \int_a^b (\|\frac{DX^{\perp}}{dt}\|^2 - K(c', X^{\perp})\|X^{\perp}\|^2)dt$.

Remark. In the case if X is collinear to c' (i.e. $X^{\perp} = 0$) we define $K(c', X^{\perp}) = 0$.

Corollary 8.3. *If $K(\Pi) < 0$ for every $p \in M$ and every 2-plane $\Pi \subset T_pM$ then every geodesic is locally minimal.*

Example 8.4. For the n-dimensional sphere S_r^n of radius r the inequality in the Bonnet – Myers Theorem becomes an equality. Hence, the bound is sharp.

Lemma 8.5. *Let $F(s, t)$ be a variation of a geodesic $c(t)$, and let $Z(s, t) \in T_{F(s,t)}M$ be smooth. Then $\frac{D}{ds}\frac{D}{dt}Z - \frac{D}{dt}\frac{D}{ds}Z = R(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t})Z$.*

Example 8.6. Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be an n-dimensional torus with arbitrary metric g (compatible with the smooth structure). Then there exists $p \in T^n$ and $v \in T_pT^n$ such that $Ric_p(v) \leq 0$.