Durham University
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Epiphany 2023

Topics in Combinatorics IV, Homework 13 (Week 13)

Due date for starred problems: **Friday, February 10, 6pm**.

13.1. (a) Let G be a simply-laced finite reflection group (i.e., all m_{ij} are equal to 2 or 3), or one of H_3 and H_4. Show that all reflections of G are conjugated in G.

(b) Let G be a finite reflection group, and let $r_1, r_2 \in G$ be two reflections. Show that the dihedral subgroup generated by r_1 and r_2 is conjugated in G to a subgroup generated by some simple reflections s_i and s_j.

13.2. (a) Let G be a Coxeter group defined by $G = \langle t_1, t_2, t_3 \mid t_i^2, (t_1t_2)^2, (t_1t_3)^2, (t_2t_3)^3 \rangle$. Show that G is isomorphic to $I_2(6)$.

(b) Show that for any odd k a Coxeter group $G = \langle t_1, t_2, t_3, \mid t_i^2, (t_1t_2)^2, (t_1t_3)^2, (t_2t_3)^k \rangle$ is isomorphic to $I_2(2k)$.

13.3. ($\star$) Let G be any group with a finite generating set S. Assume also that S is symmetric, i.e. for any $s \in S$ the inverse s^{-1} is also contained in S. Let $g, g' \in G$, and let $l(g)$ denote the shortest length of a reduced word representing g.

(a) Show that $|l(g) - l(g')| \leq l(g'g^{-1})$.

(b) Show that $d(g, g') = l(g'g^{-1})$ defines a metric on G.