Durham University
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Epiphany 2023

Topics in Combinatorics IV, Homework 15 (Week 15)

Due date for starred problems: **Friday, February 24, 6pm**.

15.1. Let $\Gamma = \langle s_1, s_2, s_3 \mid s_i^2, (s_1s_2)^3, (s_2s_3)^3, (s_1s_3)^3, s_3^7 \rangle$. Show that the subgroup generated by s_1 and s_2 is trivial, although the group $\Gamma' = \langle s_1, s_2 \mid \text{all relations not containing } s_3 \rangle$ is not.

15.2. Let (G, S) be a Coxeter system, and let $T \subset S$. Define G_T to be the subgroup of G generated by elements of T (G_T is called a *standard parabolic subgroup* of G).

(a) Let $w = s_1 \dots s_k$ be a word, all $s_i \in T$. Show that for any M-reduction $w \to w_0$ all words obtained during the procedure belong to G_T.

(b) Let $\Gamma = \langle T \mid s_i^2, (s_is_j)^{m_{ij}} \rangle$. Define a homomorphism $\varphi : \Gamma \to G$ by $\varphi(s_i) = s_i$. Show that $\ker \varphi$ is trivial.

(c) Show that (G_T, T) is a Coxeter system.

15.3. ($\star$) Let (G, S) be a Coxeter system, $s, t \in S$, and $m_{st} = \infty$ (i.e., there is no relation on st). Let w be a reduced word. Show that either $s \notin r(w)$ or $t \notin r(w)$.

15.4. Let (G, S) be Coxeter system, $r \in R$ and $g \in G$. Show that if $r \in R(g)$ then $l(rg) < l(g)$.

15.5. ($\star$) Let (G, S) be Coxeter system such that its Coxeter diagram contains a cycle. Find an element of infinite order in G.