Durham University
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Epiphany 2023

Topics in Combinatorics IV, Homework 17 (Week 17)

Due date for starred problems: **Friday, March 10, 6pm**.

17.1. ($\star$) Draw the Hasse diagram of the root poset of root system A_4.

17.2. ($\star$) Let (W,S) be a Coxeter system. A subgroup H of W is a *parabolic subgroup* if it is conjugated to a standard parabolic subgroup W_T for some $T \subset S$ (see HW 15.2), i.e. $H = w^{-1}W_T w$ for some $w \in W$. Show that for any $p \in \mathbb{R}^n$ the stabilizer $\mathrm{Stab}_W(p)$ is a parabolic subgroup.

17.3. Let (W,S) be an irreducible Coxeter system. Denote $c_n = \#\{w \in W \mid l(w) = n\}$, and define the generating function

$$W(q) = \sum_{n\geq 0} c_n q^n = \sum_{w\in W} q^{l(w)},$$

which is called the *Poincaré series* of W. In the case when W is finite, $W(q)$ is called the *Poincaré polynomial* of W.

Recall that if $T \subset S$ then W_T denotes a standard parabolic subgroup, and $W^T = \{w \in W \mid l(wt) > l(w) \forall t \in T\}$ (see HW 16.2).

For every $X \subset W$ denote also $X(q) = \sum_{w\in X} q^{l(w)}$.

(a) Show that if $T \subset S$ then $W(q) = W_T(q)W^T(q)$.

(b) Let $w \in W$, define $F = F(w) = \{s \in S \mid l(ws) > l(w)\}$. Show that $\sum_{T\subset F} (-1)^{|T|} = 0$ unless W is finite and $w = w_0$ is the longest element of W.

(c) Show that

$$\sum_{T\subset S} (-1)^{|T|} \frac{W(q)}{W_T(q)} = \sum_{T\subset S} (-1)^{|T|} W^T(q) = \begin{cases} 0 & \text{if } W \text{ is infinite,} \\ q^N & \text{if } W \text{ is finite,} \end{cases}$$

where N is the length of the longest element of W.

(d) Assume W is finite. Show that

$$\sum_{T\subset S} (-1)^{|T|} \frac{|W|}{|W_T|} = 1$$

(e) Apply the formula from (d) to compute the order of the group H_3. Can you compute the order of H_4 in this way?