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# Topics in Combinatorics IV, Homework 18 (Week 18)

**Due date** for starred problems: **Friday, March 10, 6pm**.

**18.1.** ($\star$) Let $\Delta$ be a root system, $\Pi = \{\alpha_i\}$ are simple roots.

(a) Let $\alpha, \beta \in \Delta^+$, $\alpha \lessdot \beta$ in the root poset, $\beta - \alpha = \sum\limits_{j \in J} c_j \alpha_j$ for some $J \subset [n]$. Show that either $\beta - \alpha = \alpha_j$ (i.e. $|J| = 1$ and $c_j = 1$), or $(\beta - \alpha, \alpha_j) \le 0$ for any $j \in J$.
*Hint:* Use Exercise 9.15 from lectures.

(b) Show that the root poset of a root system is ranked, where the rank of $\alpha = \sum c_i \alpha_i$ is $\sum c_i$ (this number is called the *height* of $\alpha$ and is denoted by $\operatorname{ht}\alpha$).
*Hint:* Compute $(\beta - \alpha, \beta - \alpha)$ in (a).

**18.2.** Let $\Delta$ be a root system, $\Pi = \{\alpha_i\}$ are simple roots, $W$ is the Weyl group, and $\Sigma$ is the Dynkin diagram of $\Delta$.

(a) Let $I \subset [n]$ be some index set such that $(\alpha_i, \alpha_j) = 0$ for all $i, j \in I$. Show that the standard parabolic subgroup $W_I$ is isomorphic to $(\mathbb{Z}_2)^{|I|}$.
*Hint:* use HW 15.2.

(b) Let $\alpha \in \Delta$, $\alpha = \sum c_i \alpha_i$. Define the *support* of $\alpha$ to be the set $I \subset [n]$ such that $i \in I$ if and only if $c_i \ne 0$. Show that if $(\alpha_i, \alpha_j) = 0$ for all $i, j \in I$ then $\alpha = \pm\alpha_i$ for some $i \in [n]$.

(c) Let $\alpha \in \Delta$, $\alpha = \sum c_i \alpha_i$, $I$ is the support of $\alpha$. Show that vertices of $\Sigma$ corresponding to $\alpha_i$, $i \in I$, form a connected subgraph of $\Sigma$.

**18.3.** Let $\Delta$ be a root system. Show that the highest root $\tilde{\alpha}_0$ is always long, i.e. $(\tilde{\alpha}_0, \tilde{\alpha}_0) \ge (\alpha, \alpha)$ for any $\alpha \in \Delta$.

**18.4.** Let $(W, S)$ be an irreducible Coxeter system. The goal of this exerxise is to show that if there is a quadratic form $Q$ on $\mathbb{R}^n$ invariant under $W$, then $Q(x) = c(x, x)$ for some $c \in \mathbb{R}$. Given a quadratic form $Q$, we will abuse notation by writing $Q(x, y)$ instead of $\frac{1}{2}(Q(x + y) - Q(x) - Q(y))$.

(a) Let $Q$ be a quadratic form invariant under $W$, i.e. $Q(x, y) = Q(wx, wy)$ for every $w \in W$. Recall that $Q(x, y) = (Ax, y)$ for some $A \in M_n(\mathbb{R})$. Show that $w(Ax) = A(wx)$ for every $x \in \mathbb{R}, w \in W$.

(b) For $s_i \in S$ let $s_i = r_{\alpha_i}$, $\|\alpha_i\|^2 = 2$. Show that $r_{\alpha_i}(A\alpha_i) = -A\alpha_i$, and $A\alpha_i = c_i \alpha_i$ for some $c_i \in \mathbb{R}$.

(c) Let $(s_i s_j)^2 \ne e$. Show that $c_i = c_j$. Deduce from this that $Q(x) = c(x, x)$.
*Hint*: compute $As_i(\alpha_j)$ and $s_i(A\alpha_j)$.