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Topics in Combinatorics IV, Revision problems (Week 21)

These are examples from the two revision lectures. All HW problems are also revision problems.

R.1. Write down generating functions for the numbers of Dyck paths with the leftmost peak at height (a) 1; (b) 2.

A Dyck path with the leftmost peak at height 1 starts with step up followed by a step down. Thus, there is a bijection between Dyck paths of length $2n$ with the leftmost peak at height 1 and Dyck paths of length $2n-2$: one needs to consider the interval $[2, 2n]$ of the path. Therefore, if we denote the number of Dyck paths of length $2n$ with the leftmost peak at height 1 as $F_n^{(1)}$, then $F_n^{(1)} = C_{n-1}$. In particular, there are no such paths of length 0. Thus, the generating function is

$$F^{(1)}(x) = \sum_{n\geq 1} F_n^{(1)} x^n = \sum_{n\geq 1} C_{n-1}x^n = x\sum_{n\geq 1} C_{n-1}x^{n-1} = x\sum_{k\geq 0} C_k x^k = xC(x),$$

where $C(x)$ is the generating function for the Catalan numbers.

A Dyck path with the leftmost peak at height 2 starts with two steps up followed by a step down. Thus, the number of such paths is equal to the number of lattice paths between $(3, 1)$ and $(2n, 0)$ never going below the x-axis. Adding a step up from $(2, 0)$ to $(3, 1)$, we obtain a bijection between these paths and Dyck paths of length $2n-2$ between $(2, 0)$ and $(2n, 0)$. Therefore, the generating function is again $xC(x)$.

R.2. Find the number of order-preserving bijections of the Boolean lattice B_3 to itself.

An order-preserving bijection f must preserve the set of minimal elements, so $f(\emptyset) = \emptyset$. Similarly, $f([3]) = [3]$. Also, a saturated chain should be mapped to a saturated chain, so the rank is preserved. Therefore, one-element subsets are mapped to one-element subsets. Any permutation in S_3 provides a required bijection, so the answer is 6.

R.3. Let Δ be the root system of type C_5. Let Δ_l and Δ_s be the sets of long and short roots of Δ respectively.

(a) Show that Δ_l and Δ_s are root systems and find their types.

(b) Compute the Coxeter number of Δ.

(c) Find the exponents of the Weyl group of Δ.

(a) The set of roots of Δ is $\{\pm e_i \pm e_j, \pm 2e_i\}$, where $i, j = 1, \dots, 5$, $i < j$, and $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^5$. The sets of short and long roots in a root system are invariant under reflections, so they are invariant under the subset of reflections as well. The set of long roots of Δ is $\{\pm 2e_i\}$, which compose a root system of type $(A_1)^5$. The set of short roots of Δ is $\{\pm e_i \pm e_j\}$, which compose a root system of type D_5.

(b) The number of positive roots of C_5 is $N = 25$, the rank is $n = 5$. Using the formula $h = 2N/n$ we see that $h = 10$.

Alternatively, a linear map for a Coxeter element can be written explicitly. For example, for $c = r_{e_1-e_2} r_{e_2-e_3} r_{e_3-e_4} r_{e_4-e_5} r_{2e_5}$ the matrix of C in the basis $\{e_i\}$ is

$$c = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and it is easy to see that its order is 10.

(c) The explicit expression for the matrix of a Coxeter element (see the alternative solution of part (b)) implies that the characteristic polynomial is $-x^5 - 1$, so the eigenvalues are $e^{\frac{\pi i}{5} + \frac{2k\pi i}{5}} = e^{(1+2k)\frac{2\pi i}{10}}$, where $k = 0, 1, 2, 3, 4$. Therefore, exponents are $1, 3, 5, 7, 9$.

Alternatively, one can use a result from lectures that the Young diagram $\lambda = (l_1, \dots, l_k)$ which is conjugate to the Young diagram $\mu = (m_5, m_4, m_3, m_2, m_1)$ (where m_i are the exponents) satisfies the following: l_i is equal to the number of positive roots of height i. Thus, we are left to compute the number of positive roots of every height.

A root of type $e_i - e_j$ can be written as

$$e_i - e_j = \sum_{k=i}^{j-1} e_k - e_{k+1},$$

so the height of $e_i - e_j$ is $j - i$. This number takes value 1 four times, 2 three times, 3 two times, and 4 one time.

A root of type $2e_i$ can be written as

$$2e_i = (e_i - e_5) + (e_i + e_5) = (e_i - e_5) + (e_i - e_5) + 2e_5,$$

so the height of $2e_i$ is $2(5 - i) + 1 = 11 - 2i$. This number takes values $1, 3, 5, 7, 9$ one time each.

Finally, a root of type $e_i + e_j$ can be written as

$$e_i + e_j = (e_i - e_j) + 2e_j,$$

so the height of $e_i + e_j$ is $(j - i) + 11 - 2j = 11 - (i + j)$. This number takes values $2, 3, 7$ and 8 one time each, and $4, 5, 6$ two times each.

Thus, we have $l_1 = 5$, $l_2 = l_3 = 4$, $l_4 = l_5 = 3$, $l_6 = l_7 = 2$, and $l_8 = l_9 = 1$. This implies that $\mu = (m_5, m_4, m_3, m_2, m_1) = (9, 7, 5, 3, 1)$.