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Topics in Combinatorics IV, Problems Class 5 (Week 12)

The class was devoted to the counterpart of question 11.1, where P is a regular tetrahedron in $\mathbb{R}^3$ centered at the origin.

5.1. Let P be a regular tetrahedron in $\mathbb{R}^3$ centered at the origin. A *symmetry* of P is $g \in O_3(\mathbb{R})$ taking P to itself.

(a) Show that symmetries of P compose a group, denote it by $\operatorname{Sym} P$.

If $g_1, g_2 \in O_3(\mathbb{R})$ take P to itself, then so do g_i^{-1} and $g_1 g_2$.

(b) Show that $\operatorname{Sym} P$ acts on the set of faces of P transitively.

Denote the vertices of P by $v_1, \dots, v_4$. Then $\operatorname{Sym} P$ contains rotations r_i around the lines passing through v_i and the center of the opposite face. These rotations take any face of P to any other.

(c) Show that $\operatorname{Sym} P$ acts transitively on the set of *flags*, i.e. triples (v, e, f), where v is a vertex of P, e is an edge, f is a face, and $v \in e \subset f$.

Let (v, e, f) and (v', e', f') be two flags. Using rotations r_i, we can take v to v'. Let $g_1 v = v'$, $g_1 \in \operatorname{Sym} P$, denote $e'' = g_1 e$. If $v' = v_j$, then powers of rotation r_j act transitively on all three edges incident to v_j, so there is $g_2 \in \operatorname{Sym} P$ such that $g_2(v') = v'$ and $g_2(e'') = e'$. So, $g_2 g_1$ takes v to v' and e to e', denote $f'' = g_2 g_1(f)$. If $f'' = f'$ then we are done. Otherwise, observe that there are precisely two faces containing e', and they are taken to each other by a reflection in the plane passing through e' and the center of the opposite edge. Applying this reflection, we take f'' to f', preserving e' and v'.

(d) (this is specific to cube!)

(e) Compute the order of $\operatorname{Sym} P$.

The number of flags is $4 \cdot 3 \cdot 2 = 24$, the group $\operatorname{Sym} P$ acts on the set of flags with a single orbit and trivial stabilizer. Therefore, $|\operatorname{Sym} P| = 24$.

(f) Show that $\operatorname{Sym} P$ is generated by reflections.

We have shown above that to take any flag to any flag we need rotations r_i and reflections. Now, every r_i is a product of two reflections, where the mirrors pass through the axis of r_i and any other vertex of P. There are six such reflections altogether (every mirror passes through an edge and the center of the opposite edge).

(g) Show that $\operatorname{Sym} P$ cannot be generated by two reflections.

If we assume that $\operatorname{Sym} P$ is generated by two reflections r_α and r_β, then the group would leave the space $\{\alpha, \beta\}^\perp$ of positive dimension invariant. However, $\operatorname{Sym} P$ is clearly irreducible.

Let (v, e, f) be a flag. Let $p_1 = v$, denote by p_2 the center of e, by p_3 the center of f, and by O the center of P (i.e., the origin of $\mathbb{R}^3$). Let C be the cone over triangle $p_1p_2p_3$ with apex O, i.e. the intersection of the three halfspaces: we take plane passing through $0p_ip_j$ and take the halfspace containing the third point p_k.

(h) Show that three reflections in the walls of C generate $\operatorname{Sym} P$. Write down the relations among these generators (i.e., give a presentation of $\operatorname{Sym} P$ by generators and relations, where generators are the three reflections above).

Denote by s_i the reflection in plane $0p_jp_k$. Then it is easy to see that six reflections mentioned above are s_1, s_2, s_3, $s_1s_2s_1$, $s_2s_3s_2$, and $s_1s_2s_3s_2s_1$.

We are left to find the relations, i.e. the orders of s_is_j. Reflections s_1 and s_2 generate a dihedral group preserving f, so $(s_1s_2)^3 = \mathrm{id}$. Similarly, reflections s_2 and s_3 generate a dihedral group preserving a neighboring face (which can be obtained from f by applying reflection $s_1s_2s_3s_2s_1$), so $(s_2s_3)^3 = \mathrm{id}$. Finally, the reflections s_1 and s_3 commute (as the planes $0p_1p_2$ and $0p_2p_3$ are orthogonal), so

$$\operatorname{Sym} P = \langle s_1, s_2, s_3 \mid s_i^2, (s_1s_2)^3, (s_2s_3)^3, (s_1s_3)^2 \rangle$$