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Topics in Combinatorics IV, Problems Class 7 (Week 16)

7.1. List the roots of G_2 and draw the Hasse diagram of the root poset.

The Weyl group is $I_2(6)$, so it contains six reflections. We need to express six positive roots as linear combinations of simple roots α_1 and α_2. We assume that α_1 is a short root and α_2 is long.

As $\langle \alpha_2 \mid \alpha_1 \rangle = -3$ and $\langle \alpha_1 \mid \alpha_2 \rangle = -1$, we see that

$$3 = \frac{\langle \alpha_2 \mid \alpha_1 \rangle}{\langle \alpha_1 \mid \alpha_2 \rangle} = \frac{(\alpha_2, \alpha_2)}{(\alpha_1, \alpha_1)},$$

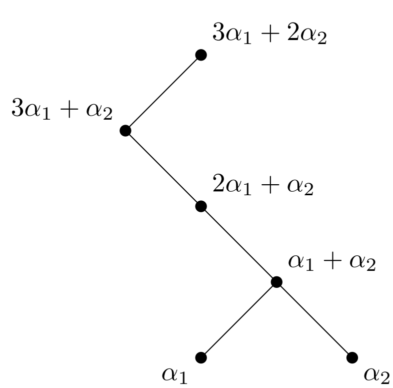
so we may assume $(\alpha_1, \alpha_1) = 1$ and $(\alpha_2, \alpha_2) = 3$. Then we have

$$(\alpha_1, \alpha_2) = \|\alpha_1\| \|\alpha_2\| (-\cos\frac{\pi}{6}) = \sqrt{3}(-\frac{\sqrt{3}}{2}) = -\frac{3}{2}$$

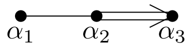
Due to Exercise 9.15, we have $\alpha_1 + \alpha_2 \in \Delta$, note that $\alpha_1 + \alpha_2 = r_{\alpha_2}(\alpha_1)$. Also, $r_{\alpha_1}(\alpha_2) = \alpha_2 - \langle \alpha_2 \mid \alpha_1 \rangle \alpha_1 = \alpha_2 + 3\alpha_1 \in \Delta$. As $(\alpha_1 + \alpha_2, \alpha_1) = 1 - \frac{3}{2} < 0$, we have $\alpha_2 + 2\alpha_1 \in \Delta$.

Finally, let us compute $(\alpha_2 + 3\alpha_1, \alpha_i)$. We have $(\alpha_2 + 3\alpha_1, \alpha_1) = -\frac{3}{2} + 3 > 0$, and $(\alpha_2 + 3\alpha_1, \alpha_2) = 3 + -3\frac{3}{2} < 0$, so $\alpha_2 + 3\alpha_1 + \alpha_2 = 2\alpha_2 + 3\alpha_1 \in \Delta$.

Computing lengths, we see that the roots $\alpha_1, \alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$ are short, and the others are long. The highest root is $2\alpha_2 + 3\alpha_1$. We get the following Hasse diagram of the root poset.



7.2. List the roots of B_3 and draw the Hasse diagram of the root poset.

The Dynkin diagram is 

, so roots α_1, α_2 are long and α_3 is short. A computation similar to the one in the G_2 case shows that we can assume $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2, (\alpha_3, \alpha_3) = 1$, $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = -1$ (and α_1 is orthogonal to α_3).

The Weyl group contains 9 reflections (cf. HW 11), so we need to express nine positive roots as linear combinations of simple roots α_1, α_2 and α_3.

We have roots simple roots $\alpha_1, \alpha_2, \alpha_3$, as well as $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ (again, due to Exercise 9.15). Now, $(\alpha_1 + \alpha_2, \alpha_3) = (\alpha_2, \alpha_3) < 0$, so $\alpha_1 + \alpha_2 + \alpha_3 \in \Delta$. Note that $\alpha_2 + \alpha_3 = r_{\alpha_2}(\alpha_3)$, so we can also find $r_{\alpha_3}(\alpha_2) = \alpha_2 + 2\alpha_3 \in \Delta$. Computing scalar product of $\alpha_2 + 2\alpha_3$ with simple roots, we see that $(\alpha_2 + 2\alpha_3, \alpha_1) = (\alpha_2, \alpha_1) < 0$, so $\alpha_1 + \alpha_2 + 2\alpha_3 \in \Delta$. Finally, we compute scalar product of $\alpha_1 + \alpha_2 + 2\alpha_3$ with simple roots and see that $(\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2) = -1 + 2 - 2 < 0$, so $\alpha_1 + 2\alpha_2 + 2\alpha_3 \in \Delta$.

The short roots are α_3 and its reflections, namely, $\alpha_2 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3$, the highest root is $\alpha_1 + 2\alpha_2 + 2\alpha_3$. We get the following Hasse diagram of the root poset.