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Topics in Combinatorics IV, Problems Class 8 (Week 18)

Problems class was mostly devoted to HW15-16. The main point is that it is a good idea to use what is known and not to use what is not known.

8.1. Let (G, S) be a Coxeter system, $w = s_{i_1} \dots s_{i_k}$ a word. Let $s_i \in S$, assume that $s_i \in R(w)$. Then $s_i = s_{i_j}$ for some j.

Suppose all letters in w are distinct from s_i. Then $s_i \in R(w)$ is a product of other generators, and thus G is generated by $S \setminus s_i$. Denote $s_j = r_{\alpha_j}$, and define $L = \operatorname{span}_{\mathbb{R}}\{\{\alpha_1, \dots, \alpha_n\} \setminus \alpha_i\}$. Then L has dimension $(n-1)$, and $\alpha_i \notin L$.

Observe that for $j \neq i$ for any $x \in \mathbb{R}^n$ we have $s_j x - x \in L$. Indeed,

$$s_j x - x = (x - \langle x \mid \alpha_j \rangle \alpha_j) - x = -\langle x \mid \alpha_j \rangle \alpha_j \in L.$$

Furthermore, for every $g \in G$ we have $gx - x \in L$. This can be proven by induction on the length of g. The base is proved above, so we need to prove that if $g'x - x \in L$ then $(s_j g')x - x \in L$. Indeed, denote $g'x - x = l \in L$. Then

$$(s_j g')x - x = s_j(g'x) - x = s_j(x + l) - x = (s_j x - x) + s_j l = (s_j x - x) + (s_j l - l) + l = l_1 + l_2 + l \in L,$$

where $l_1, l_2 \in L$. Taking $g = s_i$ and $x = \alpha_i$, we obtain $s_i \alpha_i - \alpha_i \in L$. However, $s_i \alpha_i - \alpha_i = -\alpha_i - \alpha_i = -2\alpha_i \notin L$, so we got a contradiction.