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Topics in Combinatorics IV, Terms 1[1] and 2

Contents

1 Catalan numbers

1.1 Definitions

Definition 1.1 (one of many)**.** The n-th Catalan number C_n is the number of sequences $(\varepsilon_1, \ldots, \varepsilon_{2n})$ with $\varepsilon_i = \pm 1$ such that

- $\sum_{i=1}^{2n} \varepsilon_i = 0$;
- $\sum_{i=1}^{k} \varepsilon_i \geq 0$ for every $k \leq 2n$.

Remark. Sequences in the definition above are called *ballot sequences*.

[1] Based on Alex Postnikov's notes

Example 1.2. $n = 2$: the only sequences of length $2n = 4$ are $(1, 1, -1, -1)$ and $(1, -1, 1, -1)$, so $C_2 = 2$.
$n = 3$: there are five sequences (list them!), so $C_3 = 5$.

Two equivalent definitions of C_n:

- the number of "bracketings" of a non-associative product of $n + 1$ variables;
- the number of triangulations of a convex $(n+2)$-gon on a plane (here by a triangulation we mean a maximal collection of non-crossing diagonals, it automatically subdivides the polygon into triangles).

Example 1.3. For $n = 3$, there are precisely five bracketings:

$$((a_1a_2)a_3)a_4 \quad (a_1(a_2a_3))a_4 \quad (a_1a_2)(a_3a_4) \quad a_1((a_2a_3)a_4) \quad a_1(a_2(a_3a_4)).$$

There are also precisely five triangulations of a pentagon: two non-crossing diagonals share a vertex, and there are five vertices to choose from.

Exercise 1.4. Show that the two definitions above are equivalent to Definition 1.1.

Ballot sequences can be represented by *Dyck paths*: these are paths in a $n \times n$ square going along the grid from one corner (say, $(0, n)$) to the opposite (i.e., $(n, 0)$) and staying above the main diagonal. The bijection with ballot sequences is obvious: $\varepsilon_i = 1$ becomes a move to the right, and $\varepsilon_i = -1$ becomes a move down.

One can draw the top right half of the square only, in this case a Dyck path can be understood as a path from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$ and $(1, -1)$ never going below x-axis. This is the model we will usually use in the sequel.

Remark. Paths whose all steps are vectors of $\mathbb{Z}^d$ are called *lattice paths*. The two models above for Dyck paths sit in $\mathbb{Z}^2$ generated, respectively, by $(1, 0), (0, -1)$, and $(1, 1), (1, -1)$.

Example 1.5.

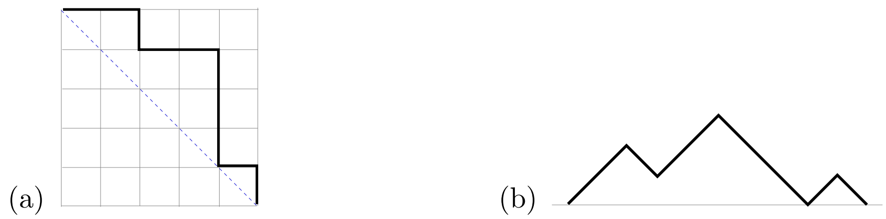


Figure 1.1: Dyck paths with steps (a) $(1,0)$ and $(0,-1)$ and (b) $(1,1)$ and $(1,-1)$.

1.2 Explicit formula for C_n

Theorem 1.6. $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

Example 1.7. $C_3 = \frac{1}{4}\binom{6}{3} = \frac{1}{4}\frac{6!}{3!3!} = \frac{6\cdot5\cdot4}{4\cdot6} = 5$; $\quad C_4 = \frac{1}{5}\binom{8}{4} = \frac{1}{5}\frac{8!}{4!4!} = \frac{8\cdot7\cdot6\cdot5}{5\cdot8\cdot3} = 14$; $\quad C_5 = \frac{1}{6}\binom{10}{5} = 42$.

We will look at three different proofs of the theorem.

Proof 1: by reflection. The number of all lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ is equal to $\binom{2n}{n}$ – choose n steps "up" out of $2n$ steps. We will compute the number of ones which go below the x-axis (call them *bad*) and subtract.

Observe that a path is bad if and only if it intersects the line $y=-1$. Find the first point where it touches the line $y=-1$, and reflect the part of the path to the right of this point with respect to the line $y=-1$. We get a new lattice path, it goes from $(0,0)$ to $(2n,-2)$. We now claim that this map establishes a bijection between *bad* paths from $(0,0)$ to $(2n,0)$ and *all* lattice paths from $(0,0)$ to $(2n,-2)$. Indeed, this map is injective, and there is an inverse: take any lattice path from $(0,0)$ to $(2n,-2)$, take the first point where it touches the line $y=-1$, and reflect the right part – we get a bad path.

Now, the paths from $(0,0)$ to $(2n,-2)$ contain $n-1$ steps up and $n+1$ steps down, so the number of all paths from $(0,0)$ to $(2n,-2)$ is $\binom{2n}{n+1}$. Thus,

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{n!n!} - \frac{n}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}$$

□

Proof 2: by cyclic shifts. First, observe that

$$\frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{2n+1}\frac{(2n+1)!}{n!(n+1)!} = \frac{1}{2n+1}\binom{2n+1}{n}.$$

Interpret $\binom{2n+1}{n}$ as all lattice paths from $(0,0)$ to $(2n+1,-1)$, i.e. paths with n steps up and $n+1$ steps down. Then C_n is the number of those paths (call them *good*) that go below the x-axis at the last step only. We want to show that good paths constitute precisely $1/(2n+1)$ part of all paths.

Let us switch to sequences of ±1. Take any sequence $(\varepsilon_1,\ldots,\varepsilon_{2n+1})$ of length $2n+1$ adding up to -1, and consider all its $2n+1$ *cyclic shifts*:

$$(\varepsilon_1,\ldots,\varepsilon_{2n+1}),\ (\varepsilon_2,\varepsilon_3,\ldots,\varepsilon_{2n+1},\varepsilon_1),\ (\varepsilon_3,\ldots,\varepsilon_{2n+1},\varepsilon_1,\varepsilon_2),\ldots,(\varepsilon_{2n+1},\varepsilon_1,\ldots,\varepsilon_{2n}).$$

Now the proof follows from the following statement which we leave as an exercise

Exercise. (1) Show that all cyclic shifts are distinct.
(2) Show that out of $2n+1$ shifts of one sequence precisely one is good. (Hint: consider the leftmost lowest point of the path).

$\square$

Definition 1.8. Let $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of non-negative integers. A formal power series $\sum_{k=0}^{\infty} a_k x^k$ is called a *generating function* of (a_n).

Example 1.9. Let $a_n = 1$ for every n. The the generating function is $A(x) = 1 + x + \cdots = 1/(1-x)$.

In general, how to find a closed formula for the generating function $A(x)$ of a sequence (a_n)? One can follow the following plan:

- write a recurrence relation on a_n;
- interpret the recurrence relation as an equation on $A(x)$;
- solve the equation and get an explicit expression for a_n.

We will proceed along this plan to get a *third proof* of the theorem.

Lemma 1.10 (Recurrence on Catalan numbers)**.**

$$C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0 = \sum_{k=1}^{n} C_{k-1}C_{n-k}$$

Proof. Take any Dyck path of length $2n$, and consider the first point $2k > 0$ where the path touches the x-axis. Then on the right there is a Dyck path of length $2n - 2k = 2(n-k)$. On the left there is a Dyck path of length $2k$ which stays strictly above x-axis, which means that we can think of it as a Dyck path of length $2k-2$ between points $(1,1)$ and $(2k-1,1)$. Both Dyck paths on the left and on the right are arbitrary, so there are precisely $C_{k-1} \cdot C_{n-k}$ Dyck paths which touch x-axis for the first time at $2k$, and the result follows.

$\square$

Defining $C_0 = 1$, we can now recursively compute any Catalan number.

Example 1.11.

$C_1 = C_0 \cdot C_0 = 1;\quad C_2 = C_0 \cdot C_1 + C_1 \cdot C_0 = 1 + 1 = 2;\quad C_3 = C_0 \cdot C_2 + C_1 \cdot C_1 + C_2 \cdot C_0 = 2 + 1 + 2 = 5.$

Lemma 1.12. *The generating function $C(x)$ satisfies the equation $xC(x)^2 - C(x) + 1 = 0$.*

Proof. We know that $C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}$. Multiplying by x^n, we get

$$C_nx^n = \sum_{k=1}^{n} C_{k-1}C_{n-k}x^n = x\sum_{k=1}^{n}(C_{k-1}x^{k-1})(C_{n-k}x^{n-k})$$

Now, summing on $n > 0$, we get on the left $\sum_{n=1}^{\infty} C_n x^n = C(x) - C_0 = C(x) - 1$. On the right, we get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} x(C_{k-1}x^{k-1})(C_{n-k}x^{n-k}) = x\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}(C_i x^i)(C_{n-i-1}x^{n-i-1}) =$$

$$= x\sum_{m=0}^{\infty}\sum_{i=0}^{m}(C_i x^i)(C_{m-i}x^{m-i}) = x \cdot C(x) \cdot C(x),$$

so we obtain the equation $C(x) - 1 = xC(x)^2$, which is precisely what we wanted.

□

Lemma 1.13.

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

Proof. Solving quadratic equation with respect to $C(x)$ (while considering x as a parameter), we see that $C(x) = \frac{1\pm\sqrt{1-4x}}{2x}$. The reason to choose the sign is the following: we want $\lim_{x\to 0} C(x) = C(0)$, where we already know $C(0) = C_0 = 1$. Now, if we consider the positive sign, then $\lim_{x\to 0} C(x) = \infty$, while the negative sign gives the required limit.

□

We are left to extract the explicit expression for C_n from the generating function. For this, we recall the definition of a generalization of binomial coefficients.

Definition 1.14. Given $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, the *binomial coefficient* $\binom{\alpha}{k}$ is given by

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}$$

Exercise. Show that

$$(1+y)^\alpha = \sum_{k\geq 0}\binom{\alpha}{k}y^k$$

We now apply the exercise above to $\alpha = 1/2$, $y = -4x$. We get

$$\sqrt{1-4x} = \sum_{k\geq 0}\binom{\frac{1}{2}}{k}(-4x)^k = 1 + \binom{\frac{1}{2}}{1}(-4x) + \binom{\frac{1}{2}}{2}(-4x)^2 + \dots,$$

so

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = -\frac{1}{2}\left(\binom{\frac{1}{2}}{1}(-4) + \binom{\frac{1}{2}}{2}(-4)^2 x + \dots\right) = -\frac{1}{2}\left(\sum_{n\geq 0}\binom{\frac{1}{2}}{n+1}(-4)^{n+1}x^n\right).$$

Therefore,

$$C_n = -\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1} = -\frac{1}{2}\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-n)}{(n+1)!}(-4)^{n+1} =$$

$$-\frac{1}{2}\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{2n-1}{2})}{(n+1)!}(-4)^{n+1} = \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{(n+1)!}\frac{2^{2n+2}}{2^{n+2}} = \frac{(2n)!}{(2^n n!)(n+1)!}\cdot 2^n = \frac{1}{n+1}\binom{2n}{n}$$

as required.

1.3 Further examples and Applications

Example 1.15 (Drunkard's walk)**.** A drunkard walks randomly along a line in steps of length 1 to the left or right with probability 1/2. The starting point is $x = 1$. The walk is terminated when the drunkard reaches the point $x = 0$. Question: what is the probability the walk terminates?

Reaching $x = 0$ after $2k + 1$ steps is equivalent to returning to $x = 1$ after $2k$ steps and then going left, so this can be understood as a Dyck path. The probability of any individual path is $(1/2)^{2k+1}$, the number of such paths of length $2k + 1$ is clearly C_k. Thus,

$$P = \sum_{k=0}^{\infty} C_k \frac{1}{2^{2k+1}} = \frac{1}{2}\sum_{k=0}^{\infty} C_k \frac{1}{2^{2k}} = \frac{1}{2}\sum_{k=0}^{\infty} C_k \left(\frac{1}{4}\right)^k = \frac{1}{2} C\left(\frac{1}{4}\right) = \frac{1}{2}\frac{1-\sqrt{1-4\frac{1}{4}}}{2\frac{1}{4}} = 1$$

We list below some further interpretations of Catalan numbers C_n.

- The number of *plane binary trees* with n vertices.

  A plane binary tree can be defined recursively: if not empty, it has a root vertex, a left subtree, and a right subtree, both of which are binary trees (either of them can be empty). When drawing such trees, the root is drawn on the top, with an edge drawn from it to the root of each of its subtrees. All binary trees with 5 vertices are shown in Fig. 1.2.

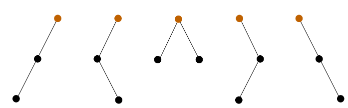


Figure 1.2: Plane binary trees with 3 vertices

- The number of *complete plane binary* trees with $n + 1$ leaves.

  A plane binary tree is complete if it has no vertices of valence 2, i.e., for every vertex both left and right subtrees are either simultaneously nonempty or empty (in the latter case the vertices are called *leaves*). All complete binary trees with 4 leaves are shown in Fig. 1.3.

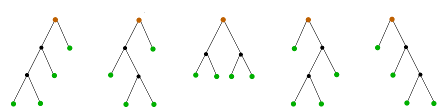


Figure 1.3: Complete plane binary trees with 4 leaves

- The number of *plane trees* with $n + 1$ vertices.

  A plane tree can also be defined recursively: it has a root vertex, and, in the case the whole tree is not a single vertex, it has a sequence $(T_1, \ldots, T_k)$ of subtrees T_i, $1 \leq i \leq k$, each of which is also a plane tree. In particular, the subtrees of each vertex are ordered; when drawing such trees, the subtrees are drawn from left to right. The root is on the top, with an edge drawn from it to the root of each of its subtrees. All plane trees with 4 vertices are shown in Fig. 1.4.

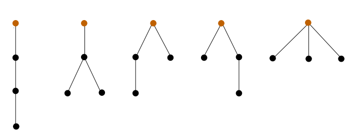


Figure 1.4: Plane trees with 4 vertices

- The number of *non-crossing matchings* on $2n$ vertices.

  A matching on $2n$ vertices can be drawn as a way to connect $2n$ nodes on the x-axis by arcs, with every node connected to precisely one other node by an arc drawn in the upper halfplane. A matching is *non-crossing* if all arcs can be drawn in a way such that no pair of arcs intersects.

  All non-crossing matchings on 6 vertices are shown in Fig. 1.5.

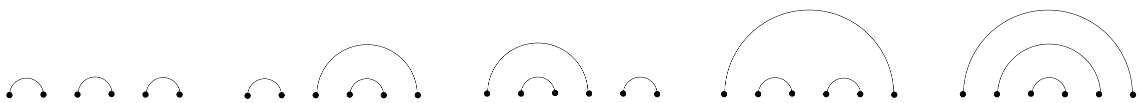


Figure 1.5: Non-crossing matchings on 6 vertices

- The number of *non-nesting matchings* on $2n$ vertices.

  A matching is *non-nesting* if all the arcs can be drawn in a way such that no arc is above the other, or, equivalently, there is no pair of arcs with ends at points (a_1, b_1) and (a_2, b_2) such that $a_1 < a_2 < b_2 < b_1$.

  All non-nesting matchings on 6 vertices are shown in Fig. 1.6.

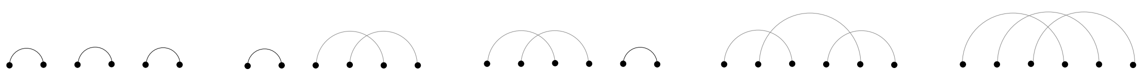


Figure 1.6: Non-nesting matchings on 6 vertices

2 Partitions and Young Diagrams

2.1 Definitions

Definition 2.1. A *partition* of $n \in \mathbb{N}$ is a sequence of integers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\sum_{i=1}^{k} \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. If $(\lambda_1, \dots, \lambda_k)$ is a partition of n, we denote $(\lambda_1, \dots, \lambda_k) \vdash n$.

Example. $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$.

Graphically, a partition can be represented by a *Young diagram* which is a left-justified array of boxes arranged in k rows, such that i-th row contains λ_i boxes, see Fig. 2.1, left, for an example.

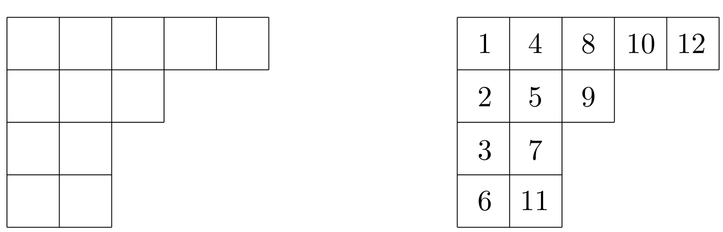


Figure 2.1: Young diagram of the partition $\lambda = (5, 3, 2, 2) \vdash 12$ and a standard Young tableau of type λ

Definition 2.2. A *standard Young tableau* (SYT for short) is a filling of a Young diagram of type $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ by numbers $1, \ldots n$ such that each number appears precisely once, and entries increase in rows (to the right) and in columns (downwards).

See Fig. 2.1, right, for an example.

Example 2.3. Let us compute the number of SYT of shape (n, n). One can assign to any such SYT a Dyck path: put a step up at every place appearing in the first row, and a step down at every place appearing in the second row. It is easy to see that this is a bijection, so the number of SYT of shape (n, n) is just the Catalan number C_n.

Denote by f_λ the number of SYT of type λ (for example, $f_{(n,n)} = C_n$ as we have seen above). Our next goal is to obtain a formula for f_λ.

2.2 The Hook Length Formula

Definition 2.4. Let (i, j) be a box in a Young diagram λ. The *hook* of (i, j) is the union of all boxes in i-th row to the right of (i, j) and all boxes in j-th column to the bottom of (i, j), i.e., $\{(i, b) \mid b \geq j\} \cup \{(a, j) \mid a \geq i\}$. The *hook length* $h(i, j)$ is the number of boxes in the hook of (i, j).

An example is shown in Fig. 2.2.

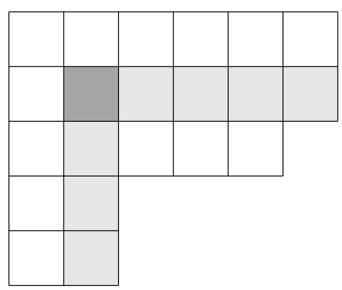


Figure 2.2: The hook of box $(2, 2)$ is shaded, $h(2, 2) = 8$

Denote $H(\lambda) = \prod\limits_{(i,j)\in\lambda} h(i, j)$.

Theorem 2.5 (The Hook Length Formula)**.** *For every partition $\lambda \vdash n$, $f_\lambda = \frac{n!}{H(\lambda)}$.*

Example 2.6.

- Let $\lambda = (4, 2, 2) \vdash 8$. Then $H(\lambda) = (6 \cdot 5 \cdot 2 \cdot 1) \cdot (3 \cdot 2) \cdot (2 \cdot 1) = 6!$, so $f_\lambda = \frac{8!}{6!} = 56$.
- Let $\lambda = (n, n)$. Then $f(1, j) = n - j + 2$ and $f(2, j) = n - j + 1$. Therefore,

$$H(\lambda) = ((n + 1) \cdot \ldots \cdot 2) \cdot (n \cdot \ldots \cdot 1) = (n + 1)!n!,$$

and thus

$$f_{(n,n)} = \frac{(2n)!}{(n + 1)!n!} = C_n$$

Let us first prove a recurrence relation on f_λ. Let λ be a Young diagram, and let c be a *corner* of λ, i.e. a box which does not have neighbors from right and bottom. Denote by $\lambda - c$ a Young diagram obtained from λ by removing the box c.

Lemma 2.7. *Define $f_\emptyset = 1$. Then*

$$f_\lambda = \sum_{c \text{ is a corner of } \lambda} f(\lambda - c)$$

Proof. Observe that n is always in the corner of any SYT of shape $\lambda \vdash n$. If c is a corner of $\lambda \vdash n$, there is a clear bijection between SYT of shape λ with n located at box c and SYT of shape $\lambda - c \vdash (n-1)$. Therefore,

$$f_\lambda = \sum_{c \text{ is a corner of } \lambda} (\text{number of SYT of shape } \lambda \text{ with } n \text{ at } c) = \sum_{c \text{ is a corner of } \lambda} f(\lambda - c)$$

as required.

□

To prove the theorem, we will use induction on the size of the Young diagram: the induction step will be to show that the numbers $n!/H(\lambda)$ satisfy the same recursion as in Lemma 2.7. The base is easy: for a one-box Young diagram, all numbers in the equation are equal to 1. Thus, we need to prove the following.

Lemma 2.8. *For every Young diagram $\lambda \vdash n$, we have*

$$\sum_{c \text{ is a corner of } \lambda} \frac{(n-1)!}{H(\lambda - c)} = \frac{n!}{H(\lambda)}$$

The Theorem 2.5 then follows: once we prove Lemma 2.8, we have

$$f_\lambda = \sum_{c \text{ is a corner of } \lambda} f(\lambda - c) = \sum_{c \text{ is a corner of } \lambda} \frac{(n-1)!}{H(\lambda - c)} = \frac{n!}{H(\lambda)},$$

where the first equality follows from Lemma 2.7, the second one is the induction assumption (applied to every corner of λ), and the last one is Lemma 2.8.

Observe that the equality in Lemma 2.8 can be reformulated as

$$\sum_{c \text{ is a corner of } \lambda} \frac{1}{n} \frac{H(\lambda)}{H(\lambda - c)} = 1$$

Therefore, we can try to interpret the summands in the left hand side as probabilities of some events. In other words, we will proceed according to the following plan: given $\lambda \vdash n$, find a random process, such that the space of outcomes is the set of corners of λ, and the probability of the outcome c is precisely $\frac{1}{n}\frac{H(\lambda)}{H(\lambda-c)}$. Then the sum of probabilities is automatically equal to 1, and thus we get the statement of the lemma.

The random process we will use is called a *hook walk* and is defined as follows.

(1) Choose randomly a box b_1 in λ, the probability of this is $p(b_1) = 1/n$.

(2) Choose randomly a box $b_2 \neq b_1$ in the hook of b_1, the probability of this is $1/(h(b_1) - 1)$.

(3) Repeat Step (2) until reach a corner.

Denote by $P(b, c)$ the probability of the walk starting at a box b ends in a corner c. The probability of an individual hook walk $b = b_1 \to b_2 \to \cdots \to b_k = c$ is equal to

$$\frac{1}{h(b_1) - 1} \cdots \frac{1}{h(b_{k-1}) - 1},$$

which implies that

$$P(b, c) = \sum_{\text{all walks } b = b_1 \to b_2 \to \cdots \to b_k = c} \frac{1}{h(b_1) - 1} \cdots \frac{1}{h(b_{k-1}) - 1}$$

Define

$$P(c) = \frac{1}{n} \sum_{b \in \lambda} P(b, c),$$

then $\sum_{c \text{ is a corner of } \lambda} P(c) = 1$. Therefore, we need to prove that

$$P(c) = \frac{1}{n} \frac{H(\lambda)}{H(\lambda - c)},$$

or, equivalently,

$$\sum_{b \in \lambda} P(b, c) = \frac{H(\lambda)}{H(\lambda - c)}$$

Example 2.9. Let $\lambda = (3, 2, 1) \vdash 6$. There are four hook paths ending in the corner located in the first row (denote it by c_1), same for the corner c_3 in the third row, and five hook paths ending in the remaining corner c_2. Then $P(c_1) = P(c_3) = 5/16$, and $P(c_2) = 3/8$.

Let us make some observations.

- Let $c = (u, v)$ be a corner. Then any hook walk ending in c is contained in the rectangle with vertices at $(1, 1), (1, v), (u, 1)$ and (u, v).
- If $c = (u, v)$ is a corner, and $p \leq u, q \leq v$, then $h(p, q) + h(u, v) = h(p, v) + h(u, q)$. Since $h(u, v) = 1$, this implies the equality
$$h(p, q) - 1 = (h(p, v) - 1) + (h(u, q) - 1)$$
Define a *weight* $\mathrm{wt}\,(p, q)$ of a box (p, q) by
$$\mathrm{wt}\,(p, q) = \frac{1}{h(p, q) - 1}$$
Then, if we denote $1/x = \mathrm{wt}\,(u, q)$ and $1/y = \mathrm{wt}\,(p, v)$, the equality above says that $\mathrm{wt}\,(p, q) = 1/(x + y)$.
- For any path, define its weight as the product of weights of all its boxes. Then we can write $P(b, c)$ as the sum of weights of all hook walks from b to c.

To complete the proof, we need further two technical lemmas.

Lemma 2.10. *Consider all lattice paths in a rectangle* $(l + 1) \times (k + 1)$ *from the top left box to the bottom right box. Denote*

$$\frac{1}{x_j} = \mathrm{wt}\,(l + 1, j), \qquad \frac{1}{y_i} = \mathrm{wt}\,(i, k + 1), \qquad x_{k+1} = y_{l+1} = 1.$$

Then

$$\sum_{\gamma \text{ is a lattice path}} \mathrm{wt}\,(\gamma) = \frac{1}{x_1 \ldots x_k y_1 \ldots y_l}$$

$\frac{1}{x_1+y_1}$	$\frac{1}{x_2+y_1}$	$\frac{1}{y_1}$	$l=1$
$\frac{1}{x_1}$	$\frac{1}{x_2}$	1	

$k=2$

Figure 2.3: Weights of boxes in a 2×3 rectangle

Example. Let $l=1, k=2$, so we consider paths in a 2×3 rectangle, see Fig. 2.3. There are $\binom{k+l}{k} = \binom{3}{2} = 3$ lattice paths from top left box to the bottom right one. Then we can compute the sum of weights of the three paths:

$$\sum_{\gamma \text{ is a lattice path}} \mathrm{wt}(\gamma) = \frac{1}{x_1+y_1} \cdot \frac{1}{x_2+y_1} \cdot \frac{1}{y_1} \cdot 1 + \frac{1}{x_1+y_1} \cdot \frac{1}{x_2+y_1} \cdot \frac{1}{x_2} \cdot 1 + \frac{1}{x_1+y_1} \cdot \frac{1}{x_1} \cdot \frac{1}{x_2} \cdot 1 = \frac{1}{x_1 x_2 y_1}$$

Proof of Lemma 2.10. We use induction on $k+l$. If $k+l=0$, then we have a 1×1 square and both sides are equal to one. To prove the induction step, observe that every path starts either from a horizontal step or from a vertical one, i.e. the second box b_2 in the path has coordinates $(1,2)$ or $(2,1)$ respectively. In both cases the remaining part of the path lies in a smaller rectangle, so we may use the induction assumption. More precisely, we have

$$\sum_{\gamma \text{ is a lattice path}} \mathrm{wt}(\gamma) = \sum_{\gamma \,|\, b_2=(1,2)} \mathrm{wt}(\gamma) + \sum_{\gamma \,|\, b_2=(2,1)} \mathrm{wt}(\gamma) =$$

$$= \frac{1}{x_1+y_1} \frac{1}{x_1 \dots x_k y_2 \dots y_l} + \frac{1}{x_1+y_1} \frac{1}{x_2 \dots x_k y_1 \dots y_l} = \frac{1}{x_1+y_1} \frac{1}{x_2 \dots x_k y_2 \dots y_l} \frac{x_1+y_1}{x_1 y_1} =$$

$$= \frac{1}{x_1 \dots x_k y_1 \dots y_l}$$

□

In the next lemma we will consider hook walks, i.e. we are allowed to miss steps in a lattice path.

Lemma 2.11. *Consider all hook walks in a rectangle $(l+1) \times (k+1)$ ending in the bottom right box. Define x_j and y_i as in Lemma 2.10. Then*

$$\sum_{\gamma \text{ is a hook walk}} \mathrm{wt}(\gamma) = \left(1 + \frac{1}{x_1}\right) \dots \left(1 + \frac{1}{x_k}\right) \left(1 + \frac{1}{y_1}\right) \dots \left(1 + \frac{1}{y_l}\right)$$

Proof. The proof follows from an easy observation: all hook walks are precisely lattice paths is some subrectangles (i.e., a rectangle constructed from boxes lying in the intersection of some of rows and some of columns of the initial rectangle). Now, choosing 1 or $1/x_j$ in the product is equivalent to the column j to be either absent or present in the subrectangle, respectively. The same holds for choosing 1 or $1/y_i$ and presence of column i. Then every such term has the form as in Lemma 2.10.

□

We can now complete the proof of Lemma 2.8. Recall that the corner c has coordinates (u, v). By Lemma 2.11, we have

$$
\begin{aligned}
P(c) = \frac{1}{n}\sum_{b\in\lambda} P(b,c) = \frac{1}{n}\sum_{\gamma \text{ is a hook walk}} \mathrm{wt}\,(\gamma) = \\
= \frac{1}{n}\left(1+\frac{1}{x_1}\right)\cdots\left(1+\frac{1}{x_k}\right)\left(1+\frac{1}{y_1}\right)\cdots\left(1+\frac{1}{y_l}\right) = \frac{1}{n}\prod_{\substack{b=(u,j)\ |\ j<v \\ b=(i,v)\ |\ i<u}}\left(1+\frac{1}{h(b)-1}\right) = \\
= \frac{1}{n}\prod_{\substack{b=(u,j)\ |\ j<v \\ b=(i,v)\ |\ i<u}}\left(\frac{h(b)}{h(b)-1}\right) = \frac{1}{n}\frac{H(\lambda)}{H(\lambda-c)},
\end{aligned}
$$

where the last equality holds since the only boxes for which the hook lengths differ in λ and $\lambda - c$ are precisely those located at row u and column v, and their hook lengths in $\lambda - c$ are one less than in λ.

2.3 Set partitions

In the previous sections we considered "unlabeled" partitions. We will now consider the "labeled" version.

Denote by $[n]$ the set of integers $1, \ldots, n$.

Definition 2.12. A *set partition* of $[n]$ is a subdivision of $[n]$ into a disjoint union of non-empty subsets (*blocks*).

Example 2.13. Let $n = 7$, we can write $[7] = \{1, 7\} \cup \{2, 3, 5\} \cup \{4, 6\}$ – this is one of partitions of shape $(3, 2, 2) \vdash 7$. Notation: if we call this partition π, we write $\pi = (1\,7 \mid 2\,3\,5 \mid 4\,6)$ (note that the order of blocks does not matter).

Graphically, we can draw set partitions as arc diagrams: vertices correspond to set elements $1, \ldots, n$, consecutive elements in one block are joined by an arc, see Fig. 2.4, left.

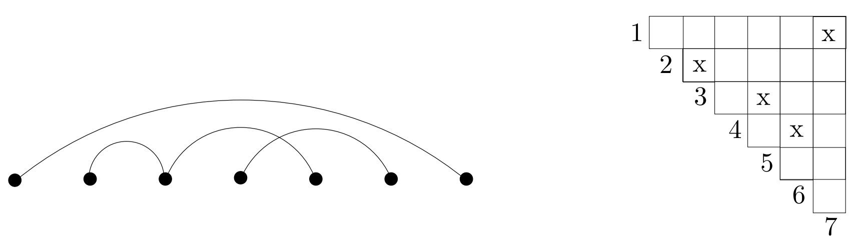


Figure 2.4: Partition $(1\,7 \mid 2\,3\,5 \mid 4\,6)$ of $[7]$: arc diagram and rook placement

The arc diagrams representing set partitions are characterized as follows: every vertex is incident to at most one arc from the left and at most one arc from the right.

Example 2.14. Another graphical interpretation of set partitions is a *non-attacking rook placement.* If we have a partition of $[n]$, consider a chess "half-board" obtained from the square $n \times n$ by taking all boxes (i, j) with $i < j$. Then place a rook in the box (i, j) if the corresponding arc diagram contains an arc connecting i and j, see Fig. 2.4, right.

Exercise. Show that rooks placed as described above do not attack each other. Show that this map has an inverse: any non-attacking rook placement gives rise to a set partition.

Note that the number of rooks (which is also the number of arcs) is equal to $n-k$, where k is the number of blocks.

Definition 2.15. The number of set partitions of $[n]$ is called *Bell number* $B(n)$. The number of set partitions of $[n]$ into k blocks is called *Stirling number of second kind* $S(n,k)$, i.e. $B(n) = \sum_{k=1}^{n} S(n,k)$.

Example 2.16. There are five set partitions of [3], namely

$$(1 \mid 2 \mid 3), \quad (1\,2 \mid 3), \quad (1 \mid 2\,3), \quad (1\,3 \mid 2), \quad (1\,2\,3),$$

so $B(3) = 5$, $S(3,1) = S(3,3) = 1$, $S(3,2) = 3$.

Definition 2.17. A set partition is *non-crossing* if the arc diagram is non-crossing. A set partition is *non-nesting* if the arc diagram is non-nesting.

Example 2.18. For $n = 1,2,3$ all set partitions are non-crossing and non-nesting. For $n = 4$, there is precisely one set partition which is not non-crossing and one which is not non-nesting.

Exercise. Compute $B(4)$ by listing all arc diagrams.

Note that arc diagrams of non-crossing (non-nesting) partitions are not the same as of non-crossing (non-nesting) matchings. However, the following result holds.

Theorem 2.19. *The number of non-crossing set partitions of* $[n]$ *is equal to the number of non-nesting set partitions of* $[n]$ *and is equal to the Catalan number* C_n.

Definition 2.20. Given a Dyck path, a *peak* is a local maximum (i.e., a step up followed by a step down), and a *valley* is a local minimum (i.e., a step down followed by a step up). Clearly, the number of peaks exceeds the number of valleys by one.

Proof of Theorem 2.19. First we construct a bijection between non-nesting set partitions of $[n]$ and Dyck paths of length $2n$. A partition is not non-nesting if and only if there are two arcs (a_i, b_i) such that $a_i < a_j < b_j < b_i$. This is equivalent to the rook (a_j, b_j) in the corresponding rook placement being located in the positive octant with respect to the origin at (a_i, b_i) (here by "positive" we mean standard coordinates in $\mathbb{R}^2$). Therefore, the equivalent criterion for a set partition being non-nesting is the following: the positive octant with respect to every rook does not contain any other rook.

Now we can construct the map: take a set partition, consider the corresponding rook placement, draw the union of all positive octants centered at rooks, and then the boundary of this domain will be a Dyck path. The inverse map is constructed as follows: put rooks in all valleys of a Dyck path.

Given a Dyck path, a non-crossing partition can be constructed via a "shelling algorithm", see Fig. 2.5 for an example.

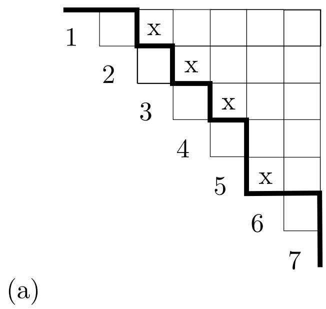
□

Exercise 2.21. Fill in the details of the proof.

Definition 2.22. *Narayana number* $N(n,k)$ is the number of Dyck paths of length $2n$ with k peaks.

Corollary 2.23 (of the proof of Thm. 2.19)**.** *The number of non-nesting partitions of* $[n]$ *with* k *blocks is equal to* $N(n, n-(k-1))$.

Example 2.24. Let us compute $N(4,2)$, i.e. the number of Dyck paths of length 8 with 2 peaks. By Cor. 2.23, $N(4,2)$ is equal to the number of non-nesting partitions of [4] with 3 blocks, which, in its turn, is equal to the number of rook placements, where the number of rooks is equal to $4-3=1$. The number of boxes in the upper half of the 4×4 board is equal to 6, so there are 6 ways to place one rook, and thus $N(4,2) = 6$.



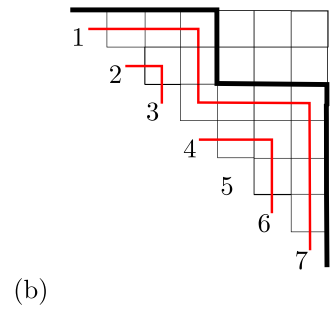


Figure 2.5: (a) non-nesting partition $(1\,3\,5\,6 \mid 2\,4 \mid 7)$ of $[7]$: rook placement and Dyck path; (b) non-crossing partition $(1\,7 \mid 2\,3 \mid 4\,6 \mid 5)$ of $[7]$ and its Dyck path

2.4 Generating functions

Recall: if we have a sequence (a_n), we can define a generating function of (a_n), which is a formal power series $a_0 + a_1x + a_2x^2 + \dots$. This is an *ordinary* generating function. We can also define an *exponential* generating function.

Definition 2.25. *Exponential generating function* of a sequence $(a_n)_{n\geq o}$ is a formal power series $\sum\limits_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

Usually, ordinary generating functions are used when the objects are unlabeled, and exponential generating functions are used when the objects are labeled.

Example 2.26. The number p_n counts *integer partitions* of n, i.e. the number of Young diagrams $\lambda \vdash n$. Bell number $B(n)$ counts partitions of $[n]$. Bell numbers are "labeled versions" of $p(n)$, so it is reasonable to look for an ordinary generating function for (p_n) and an exponential generating function for (B_n).

Our next goal is to write the generating functions for $p(n)$ and $B(n)$.

2.4.1 Integer partitions

Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$ be a partition. We can write it as $\lambda = 1^{m_1}2^{m_2}3^{m_3}\dots$, where m_i is the number of $\lambda_j = i$.

Example. $\lambda = (4, 2, 1, 1, 1) \vdash 9$ can be written as $\lambda = 1^3 2^1 3^0 4^1$ (formally, there are infinitely many zero powers at the end, but we omit all factors r^0 for $r > \lambda_1$).

The notation above provides a bijection between all partitions of natural numbers and all sequences $(m_i)_{i>0}$ of non-negative integers with finite number of positive entries. Given a sequence (m_i), it corresponds to a partition of $n = 1 \cdot m_1 + 2 \cdot m_2 + 3 \cdot m_3 \dots$. If $\lambda \vdash n$, denote $|\lambda| = n$.

Then the (ordinary) generating function of $p(n)$ is

$$\sum_{n\geq 0} p(n)x^n = \sum_{n\geq 0} \underbrace{x^n + x^n + \dots + x^n}_{p(n) \text{ summands}} = \sum_{n\geq 0,\, \lambda\vdash n} x^{|\lambda|} = \sum_{\lambda \text{ any partition}} x^{|\lambda|} =$$

$$= \sum_{m_1, m_2, \dots \geq 0} x^{1\cdot m_1 + 2\cdot m_2 + 3\cdot m_3 + \dots} = \left(\sum_{m_1\geq 0} x^{m_1}\right)\left(\sum_{m_2\geq 0} x^{2m_2}\right)\left(\sum_{m_3\geq 0} x^{3m_3}\right)\dots =$$

$$= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\dots = \prod_{k\geq 1}\frac{1}{1-x^k}$$

Note that although the product is infinite, the number of terms contributing to any coefficient is finite.

2.4.2 The exponential formula

Let numbers c_n count some objects on n labeled nodes (call them *c-objects*), where $n > 0$. Let $c(x) = \sum_{n\geq 1} c_n \frac{x^n}{n!}$ be the exponential generating function of (c_n).

Define *d-objects* on n nodes as collections of c-objects on $n_1, \ldots, n_k$ nodes such that $n_1 + \cdots + n_k = n$. Let $d(x) = \sum_{n\geq 0} d_n \frac{x^n}{n!}$ be the exponential generating function of (d_n).

Example 2.27. Let c-objects be linear graphs on n nodes (or, equivalently, sequences of n distinct numbers $1, \ldots, n$). Then $c_n = n!$, and thus $c(x) = x + x^2 + \cdots = \frac{x}{1-x}$.

Then d-objects are graphs on n nodes, such that any connected component is a linear graph (or, equivalently, unions of sequences of total length n). How to compute $d(x)$?

Lemma 2.28 (Exponential formula)**.**

$$d(x) = e^{c(x)}$$

Proof. A d-object on n nodes consists of a set partition of $[n]$, $n = n_1 + \cdots + n_k$, and then a c-object on every block. Let us fix k and $n_1, \ldots, n_k$, and compute d-objects.

The number of ways to subdivide n into k blocks of size $n_1, \ldots, n_k$ is equal to the multinomial coefficient

$$\binom{n}{n_1\, n_2\, \ldots\, n_k} = \frac{n!}{n_1!\, n_2!\, \ldots\, n_k!}$$

We have a c-object on every block, so the number of d-object with fixed k and $n_1, \ldots, n_k$ is equal to

$$\frac{1}{k!}\binom{n}{n_1\, n_2\, \ldots\, n_k} c_{n_1} \ldots c_{n_k},$$

where we divide by $k!$ as we are not interested in the order of the blocks.

Now take a sum over $k \geq 0$ and $n_1 + \cdots + n_k = n$:

$$d_n = \sum_{k\geq 0} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\ldots,n_k\geq 1}} \frac{1}{k!}\frac{n!}{n_1!\, n_2!\, \ldots\, n_k!} c_{n_1} \ldots c_{n_k},$$

which implies that

$$\frac{d_n}{n!} = \sum_{k\geq 0} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\ldots,n_k\geq 1}} \frac{1}{k!}\frac{c_{n_1}}{n_1!}\frac{c_{n_2}}{n_2!}\cdots\frac{c_{n_k}}{n_k!},$$

and thus

$$d(x) = \sum_{n\geq 0} d_n \frac{x^n}{n!} = \sum_{n\geq 0}\sum_{k\geq 0} \sum_{\substack{n_1+\cdots+n_k=n \\ n_1,\ldots,n_k\geq 1}} \frac{1}{k!}\frac{c_{n_1}}{n_1!}\frac{c_{n_2}}{n_2!}\cdots\frac{c_{n_k}}{n_k!} x^n =$$

$$= \sum_{k\geq 0}\frac{1}{k!} \sum_{n_1,\ldots,n_k\geq 1} \frac{c_{n_1}x^{n_1}}{n_1!}\frac{c_{n_2}x^{n_2}}{n_2!}\cdots\frac{c_{n_k}x^{n_k}}{n_k!} = \sum_{k\geq 0}\frac{1}{k!}\left(\sum_{m\geq 1}\frac{c_m x^m}{m!}\right)^k = \sum_{k\geq 0}\frac{1}{k!} c(x)^k = e^{c(x)}.$$

$\square$

Corollary 2.29. *The exponential generating function for Bell numbers is*

$$B(x) = e^{e^x - 1}$$

Proof. Let c-objects be just sets (so, $c_n = 1$), and d-objects be collections of sets of total cardinality n. Then d-objects are precisely partitions of $[n]$. Since $c_n = 1$, we have $c(x) = e^x - 1$, and thus $B(x) = d(x) = e^{e^x - 1}$ by the exponential formula.

□

3 Permutations

3.1 Definitions and notation

Recall that a *permutation* is a bijection $w : [n] \to [n]$. Permutations form a group (called *symmetric group*) S_n with respect to composition, the order of the group is $n!$.

There are several notations for $w \in S_n$.

- 2-*line notation*: if $w(i) = w_i$, we write

$$w = \begin{pmatrix} 1 & 2 & \dots & n \\ w_1 & w_2 & \dots & w_n \end{pmatrix}, \qquad \text{e.g.} \qquad w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 6 & 1 & 5 & 7 & 4 \end{pmatrix} \in S_7;$$

- 1-*line notation*: we simply write $w = w_1, w_2, \dots, w_n$, or even $w = w_1\, w_2\, \dots\, w_n$, e.g. $w = 3\,2\,6\,1\,5\,7\,4$;
- *cycle notation*: recall that every permutation can be decomposed a a product of disjoint cycles. For the permutation above, we have $w = (1\ 3\ 6\ 7\ 4)(2)(5)$. Note that the order of cycles is irrelevant, as well as the starting point of each cycle: we could also write $w = (2)(6\ 7\ 4\ 1\ 3)(5)$.

3.2 Statistics on permutations

Definition 3.1. A *statistics* on permutations is a function $\sigma : S_n \to \mathbb{Z}_{\geq 0}$. Its *generating function* is a polynomial $f_\sigma(x) = \sum\limits_{w \in S_n} x^{\sigma(w)}$.

Two statistics σ and μ are *equidistributed* if $f_\sigma(x) = f_\mu(x)$, i.e. the number of permutations with the same value of statistics is the same.

Definition 3.2. Let $w \in S_n$.

- An *inversion* is a pair (i, j) such that $i < j$, $w_i > w_j$;
- A *descent* is $i \in [n-1]$ such that $w_{i+1} < w_i$.

We can now look at some statistics on S_n.

- $\operatorname{inv}(w)$ is the number of inversions in w;
- $\operatorname{des}(w)$ is the number of descents in w;
- $\operatorname{cyc}(w)$ is the number of cycles in w.

Example 3.3. Let $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$. Then $\operatorname{inv}(w) = 8$, $\operatorname{des}(w) = 3$, $\operatorname{cyc}(w) = 3$.

Definition 3.4. A statistics on S_n is called *Mahonian* if it is equidistributed with inv, and *Eulerian* if it is equidistributed with des.

Example 3.5. Let $n = 3$, order permutations by the number of inversions. Then we have the following.

w	123	213	132	231	312	321
$\operatorname{inv}(w)$	0	1	1	2	2	3
$\operatorname{des}(w)$	0	1	1	1	1	2
$\operatorname{cyc}(w)$	3	2	2	1	1	2

In particular, we can see that all three statistics are different.

We can also compute the generating functions:

$$\begin{aligned} f_{\mathrm{inv}}(x) &= 1\cdot x^0 + 2\cdot x^1 + 2\cdot x^2 + 1\cdot x^3 = (1+x)(1+x+x^2) \\ f_{\mathrm{des}}(x) &= 1\cdot x^0 + 4\cdot x^1 + 1\cdot x^2 = 1 + 4x + x^2 \\ f_{\mathrm{cyc}}(x) &= 0\cdot x^0 + 2\cdot x^1 + 3\cdot x^2 + 1\cdot x^3 = x(x+1)(x+2) \end{aligned}$$

Theorem 3.6. *For every $n \in \mathbb{N}$,*

$$f_{\mathrm{inv}}(x) = (1+x)(1+x+x^2)\dots(1+x+\dots+x^{n-1}) = \frac{\prod_{k=1}^{n}(1-x^k)}{(1-x)^n}$$

Proof. We use induction on n. For $n = 1$ the result is clear (as for $n = 2, 3$). Assume that the theorem holds for $n-1$. Take any permutation $w' \in S_{n-1}$ and "insert" n in all possible n places. Depending on the place where n is inserted, we add from 0 to $n-1$ inversions, where all numbers show up. Therefore, the generating function is

$$f_{\mathrm{inv}}(x) = \sum_{w \in S_n} x^{\operatorname{inv}(w)} = \sum_{w' \in S_{n-1}} x^{\operatorname{inv}(w')} + x \sum_{w' \in S_{n-1}} x^{\operatorname{inv}(w')} + x^2 \sum_{w' \in S_{n-1}} x^{\operatorname{inv}(w')} + \dots + x^{n-1} \sum_{w' \in S_{n-1}} x^{\operatorname{inv}(w')} =$$

$$= \left(\sum_{w' \in S_{n-1}} x^{\operatorname{inv}(w')}\right)(1+x+\dots+x^{n-1}) = (1+x)(1+x+x^2)\dots(1+x+\dots+x^{n-1}),$$

where the last equality follows from the induction assumption.

□

Definition 3.7. The *Major index* of $w \in S_n$ is $\operatorname{maj}(w) = \sum_{k \text{ is a descent of } w} k$.

Example 3.8. For $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$, we have $\operatorname{maj}(w) = 1 + 3 + 6 = 10$.

For $n = 3$, we have

w	123	213	132	231	312	321
$\operatorname{maj}(w)$	0	1	2	2	1	3

and thus $f_{\mathrm{maj}}(x) = 1 + 2x + 2x^2 + x^3 = (1+x)(1+x+x^2) = f_{\mathrm{inv}}(x)$.

Exercise. Show that $f_{\mathrm{maj}}(x) = f_{\mathrm{inv}}(x)$ for every $n \in \mathbb{N}$, i.e. maj is Mahonian.

Definition 3.9. Let $w = w_1 w_2 \dots w_n \in S_n$, w_i is a *record* of w if $w_i > w_j$ for all $j < i$. Denote by $\operatorname{rec}(w)$ the number of records of w.

Example. for $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$, we have $\mathrm{rec}\,(w) = 3$.

For $n = 3$, we have

w	$1\,2\,3$	$2\,1\,3$	$1\,3\,2$	$2\,3\,1$	$3\,1\,2$	$3\,2\,1$
$\mathrm{rec}\,(w)$	3	2	2	2	1	1

and thus $f_{\mathrm{rec}}\,(x) = 2x + 3x^2 + x^3 = f_{\mathrm{cyc}}\,(x)$.

Theorem 3.10. *$f_{\mathrm{rec}}\,(x) = f_{\mathrm{cyc}}\,(x)$ for every $n \in \mathbb{N}$.*

Proof. We construct a bijection $f : S_n \to S_n$, such that if $w' = f(w)$ then $\mathrm{rec}\,(w') = \mathrm{cyc}\,(w)$.

First, write the cycle decomposition in a standard way: if $w = (a_1 \dots)(a_2 \dots) \dots (a_k \dots)$ then every a_i is maximal in its cycle, and $a_1 < a_2 < \cdots < a_k$.

Now, the map f takes w written in the standard way above to w' by erasing all brackets (and considering the result as a 1-line notation for w'). Clearly, records are first elements of cycles, so $\mathrm{rec}\,(w') = \mathrm{cyc}\,(w)$.

The map f has an inverse: take any permutation w' in 1-line notation, put brackets in the beginning and at the end, and then put closing and opening brackets before every record. We get a cycle decomposition of some permutation w. Then $f(w) = w'$.

□

Example 3.11. The preimage of $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$ under f is $u = (3\,2\,6)(1\,5)(7\,4)$.

Now, $w = 3\,2\,6\,1\,5\,7\,4 = (1\,3\,6\,7\,4)(2)(5) = (2)(5)(7\,4\,1\,3\,6)$, so $f(w) = 2\,5\,7\,4\,1\,3\,6$.

Definition 3.12. Let $w = w_1\,w_2\,\dots\,w_n \in S_n$, $i \in [n]$ is an *excedance* of w if $i < w_i$. Denote by $\mathrm{exc}\,(w)$ the number of excedances of w.

Example. For $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$, we have $\mathrm{exc}\,(w) = 3$.

For $n = 3$, we have

w	$1\,2\,3$	$2\,1\,3$	$1\,3\,2$	$2\,3\,1$	$3\,1\,2$	$3\,2\,1$
$\mathrm{exc}\,(w)$	0	1	1	2	1	1

and thus $f_{\mathrm{rec}}\,(x) = 1 + 4x + x^2 = f_{\mathrm{des}}\,(x)$.

Theorem 3.13. *$f_{\mathrm{exc}}\,(x) = f_{\mathrm{des}}\,(x)$ for every $n \in \mathbb{N}$, so* exc *is Eulerian.*

Proof. Define *anti-excedance* of w as $i \in [n]$ such that $i > w_i$. Observe, that anti-excedances of w are precisely excedances of w^{-1}, so anti-excedances and excedances are equidistributed.

Now, we claim that the map f from the proof of Thm 3.10 takes anti-excedances of w to descents of $w' = f(w)$.

□

Exercise. Complete the proof of the theorem.

4 Posets and lattices

4.1 Definitions

Definition 4.1. A *partially ordered set*, or *poset*, P is a set with a binary relation $\leq$ (or $\leq_P$ if there is an ambiguity) satisfying the following axioms:

- $a \leq a$ (reflexivity);

· if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity);

· if $a \leq b$ and $b \leq a$ then $a = b$ (symmetry).

We say $a < b$ if $a \leq b$ and $a \neq b$, and the notation $a \geq b$ means that $b \leq a$.
Two elements are *incomparable* if $a \not\leq b$ and $b \not\leq a$.

Example 4.2. Any ordered field (e.g. $\mathbb{R}$) is a poset, where one considers usual order relation. (Such posets are *totally ordered*, i.e. they do not contain incomparable elements).

The power set of any set (i.e., the set of all subsets), with the order being inclusion, is also a poset (see below). Two subsets are incomparable if their symmetric difference is not empty.

Definition 4.3. A *covering relation* $\lessdot$ in a poset P is defined as follows: $a \lessdot b$ if $a < b$ and there is no element $c \in P$ such that $a < c < b$.

Definition 4.4. A *Hasse diagram* of a poset P is an oriented graph, where vertices are elements of P, and there is an edge $a \to b$ if $a \lessdot b$. Hasse diagram is usually drawn as an undirected graph with b positioned above a if $a \lessdot b$.

Note that any finite poset is uniquely defined by its Hasse diagram.

Example 4.5. The Hasse diagram shown in Fig. 4.1 defines a poset. We have $x \leq y$ if and only if there is a path from x to y every step of which goes up. For example, $a \leq c, d, e, f$ and is not comparable to b. Overall, there are five pairs of incomparable elements.

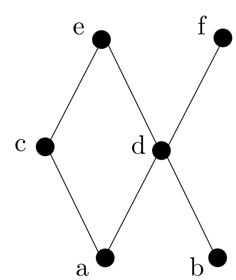


Figure 4.1: Hasse diagram of a poset.

Definition 4.6. A *chain* in a poset P is a sequence of elements $a_1 < a_2 < \cdots < a_k$. A chain is *saturated* if $a_i \lessdot a_{i+1}$ for every $i = 1, \ldots, k-1$. An *antichain* is a subset of P consisting of mutually incomparable elements.

Example. In the poset defined in Fig. 4.1, (c, f) is an antichain, and (a, d, f) is a chain as well as (b, e).

Definition 4.7. A *Boolean lattice* B_n is the poset of subsets of $[n]$ ordered by inclusion: $A \leq B$ if $A \subset B$. The set of subsets (or the power set) of $[n]$ is denoted by $2^{[n]}$.

Example 4.8. Hasse diagrams for B_2 and B_3 are shown in Fig. 4.2. Note that for every n the Hasse diagram for B_n is a 1-skeleton of a cube.

We are interested in the following questions: what are the maximal sizes of chains and antichains in posets? For example, in the poset defined in Fig. 4.1, these are 3 and 2 respectively. In B_n, the maximal size of chain is $n + 1$, but what is the maximal size of an antichain?

Figure 4.2: Hasse diagram of B_2 and B_3.

4.2 Sperner's Theorem

In this section we will answer the question above by proving the following result.

Theorem 4.9 (Sperner's Theorem)**.** *Let $S_1, \dots, S_N$ be different subsets of $[n]$ such that $S_i \not\subset S_j$ and $S_j \not\subset S_i$ for any $i \neq j$. Then $N \leq \binom{n}{\lfloor n/2 \rfloor}$, where $\lfloor x \rfloor$ is the maximal integer not exceeding x.*

Remark. Note that the bound in the theorem is sharp: the number of $\lfloor n/2 \rfloor$-subsets of $[n]$ is precisely $\binom{n}{\lfloor n/2 \rfloor}$, and they are all incomparable.

Definition 4.10. A poset P is *ranked* if there is a function $\rho : P \to \mathbb{Z}$ such that $a \lessdot b$ implies $\rho(b) = \rho(a) + 1$. If P is finite, we assume that the minimal rank is zero. If P is finite and ranked, the *rank numbers* r_i are defined as the numbers of elements of P of rank i.

In other words, a poset is ranked if its Hasse diagram can be drawn "on levels".

Example 4.11. The left poset shown in Fig. 4.3 is ranked, and the right is not.

Figure 4.3: Ranked (left) and not ranked (right) posets.

A finite ranked poset P with non-zero rank numbers $r_0, \dots, r_l$ is *rank symmetric* if $r_i = r_{l-i}$, *unimodal* if there is k such that $r_0 \leq r_1 \leq r_k \geq r_{k+1} \geq \dots \geq r_l$, and *Sperner* if the maximal size of antichain is the maximum of r_i (i.e., the maximal size of antichain is minimal possible).

Example 4.12. The Hasse diagram in Fig. 4.4 defines a non-Sperner poset: both rank numbers are four, but there is an antichain of size five.

Remark. The Boolean lattice B_n is ranked, where $\rho(A) = |A|$, with ranking numbers $r_i = r_i(B_n) = \binom{n}{i}$. Since $\binom{n}{i} = \binom{n}{n-i}$, B_n is rank-symmetric. It is also unimodal as

$$\binom{n}{0} \leq \binom{n}{1} \leq \dots \leq \binom{n}{\lfloor n/2 \rfloor} \geq \dots \geq \binom{n}{n}$$

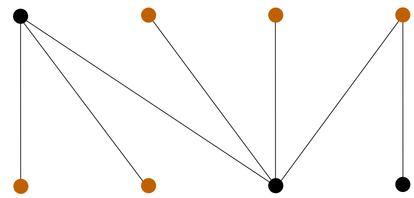


Figure 4.4: Non-Sperner rank-symmetric poset. Colored vertices form an antichain of size 5.

Then Theorem 4.9 can be reformulated as: B_n is Sperner.

Definition 4.13. Let P be a finite ranked poset, ρ is the ranking function with values $0, \dots, l$. A *symmetric chain decomposition* (or SCD for short) of P is a decomposition of P as a disjoint union $P = C_1 \sqcup C_2 \sqcup \dots \sqcup C_k$, such that each C_i is a saturated chain, and if $C_i = a_0^i \lessdot a_1^i \lessdot \dots \lessdot a_s^i$ then $\rho(a_0^i) + \rho(a_s^i) = l$.

Example. B_2 has a SCD: $C_1 = \emptyset \lessdot \{1\} \lessdot \{1,2\}$ and $C_2 = \{1\}$.

Lemma 4.14. *If a finite ranked poset P has a SCD, then it is rank-symmetric, unimodal and Sperner.*

Proof. P is clearly rank-symmetric and unimodal, where the maximal ranking number is r_i for $i = \lfloor l/2 \rfloor$, as this property holds for every chain in the SCD. We need to show that P is Sperner.

Take any antichain A, then $|A \cap C_i| \leq 1$, so $|A| \leq k$. Since every chain C_i intersects the middle level, the number of chains $k = r_{\lfloor l/2 \rfloor}$, so the size of any antichain does not exceed the maximal ranking number, which is precisely the definition of Sperner poset. $\square$

Due to Lemma 4.14, to prove Thm 4.9 we are left to show that B_n has a SCD.

Definition 4.15. Let P_1, P_2 be posets, then $P_1 \times P_2$ is also a poset: we can define the order by $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. Similarly, one can define a product $P_1 \times \dots \times P_n$ of any number of posets.

Observe that $B_n = [2]^n = [2] \times \dots \times [2]$. Indeed, we can assign to every element A of B_n a sequence of 0 and 1 of length n, where $a_i = 0$ if $i \notin A$ and $a_i = 1$ if $i \in A$. Then $A \subset B$ is equivalent to $a_i \leq b_i$ for all i, and thus to $A \leq B$.

We are now left to prove the following statement.

Theorem 4.16. *Any product of chains has a SCD.*

Proof. We proceed by induction. First, a product of two chains has a SCD (prove this!). Now take $P = P' \times C = (C_1 \sqcup C_2 \sqcup \dots \sqcup C_k) \times C$. As a product of two chains has a SCD, every $C_i \times C$ has a SCD. Note that the "middle rank" of all $C_i \times C$ is the same – this is guaranteed by the fact $P' = C_1 \sqcup C_2 \sqcup \dots \sqcup C_k$ is a SCD. Thus, taking the union of all SCD for $C_i \times C$, $i \in [k]$, we obtain a SCD for P.

$\square$

4.3 Greene's Theorem

Let P be a finite poset.

Theorem 4.17 (Dilworth)**.** *The maximal size of an antichain of P is equal to the minimal number of chains needed to cover P.*

Remark. Note that one inequality in the theorem is evident: an antichain cannot contain more elements than the number of chains as it intersect every chain at most once.

A dual version also holds.

Theorem 4.18 (Minsky)**.** *The maximal size of a chain of P is equal to the minimal number of antichains needed to cover P.*

More generally, define l_k to be the maximal size of a union of k chains of P, and m_k to be the maximal size of a union of k antichains of P. In particular, $l_0 = m_0 = 0$, and l_1 is the maximal size of a chain in P.

Theorem 4.19 (Greene)**.** *Denote $\lambda_i = l_i - l_{i-1}$ for $i > 0$, and let $\lambda(p) = \lambda_2, \lambda_2, \dots$ (note that only finitely many λ_i are positive). Similarly, denote $\mu_i = m_i - m_{i-1}$ for $i > 0$, and let $\mu(p) = \mu_2, \mu_2, \dots$. Then $\lambda_i \geq \lambda_{i+1}$ and $\mu_i \geq \mu_{i+1}$ for all $i > 0$ (so λ and μ can be considered as Young diagrams or partitions of $|P|$), and Young diagrams λ and μ are* conjugate *to each other, i.e. they are symmetric with respect to the main diagonal.*

Example. Let P be defined by the Hasse diagram shown in Fig. 4.5. Then $l_1 = 3$, $l_2 = 5$, and $l_i = 5$ for $i \geq 2$. Also, $m_1 = 2$, $m_2 = 4$, $m_3 = 5$, and $m_i = 5$ for $i \geq 3$. Therefore, $\lambda = (3, 2)$, and $\mu = (2, 2, 1)$ which are clearly conjugated.

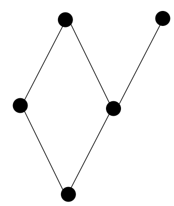


Figure 4.5: Hasse diagram of a poset.

Remark. Theorems of Dilworth and Minsky are partial cases of Greene's Theorem. Indeed, the maximal size of antichain is $m_1 = \mu_1$, i.e. the length of the first row of μ. The minimal number of chains needed to cover P is precisely the number of non-zero λ_i, i.e. the number of rows of λ. Greene's Theorem says that Young diagrams λ and μ conjugate, which implies that the first row of μ is equal to the first column of λ, and this is precisely Dilworth's Theorem. Minsky's Theorem follows similarly.

Given $w \in S_n$, define a poset P_w as follows: elements of P_w are elements of $[n]$, and $w_i <_{P_w} w_j$ if $w_i < w_j$ and $i < j$.

Example. Let $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$. Hasse diagram of P_w is shown in Fig. 4.6.

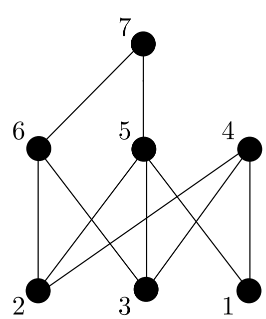


Figure 4.6: Hasse diagram of the poset P_w for $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$.

Exercise. Show that chains of P_w are precisely increasing subsequences of w_i, and antichains are decreasing subsequences of w_i.

Another corollary of Greene's Theorem is the following statement.

Theorem 4.20 (Erdos–Szekeres)**.** *Let $m, n \geq 1$. Then any permutation of size at least $mn+1$ contains either an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$.*

Proof. According to Greene's Theorem, we can associate a Young diagram λ to P_w, in which the length of the first row is the maximal size of chain in P_w, i.e. the maximal length of an increasing subsequence in w (see the exercise above). Similarly, the length of the first row of the conjugate Young diagram (and thus the length of the first column of λ) is the maximal size of antichain in P_w, i.e. the maximal length of an decreasing subsequence in w. The rest follows from the following elementary statement.

Exercise. Let λ be a Young diagram, $|\lambda| \geq mn+1$. Then either $\lambda_1 \geq n+1$, or $\lambda'_1 \geq m+1$, where λ'_1 is the length of the first column of λ.

$\square$

Exercise. Prove Theorem 4.20 without using Greene's Theorem.

4.4 Lattices

Definition 4.21. Let P be a poset, $x, y \in P$.

- $z \in P$ is a *join* of x and y (notation $z = x \vee y$) if the following hold:

  (1) $z \geq x, y$;

  (2) if $v \in P$ and $v \geq x, y$ then $v \geq z$.

  Note that (2) implies that if join exists then it is unique.

- $t \in P$ is a *meet* of x and y (notation $t = x \wedge y$) if the following hold:

  (1) $t \leq x, y$;

  (2) if $v \in P$ and $v \leq x, y$ then $v \leq t$.

  Again, meet is unique if exists.

- P is a *lattice* if every two elements have a join and a meet.

Example. Out of the three posets whose Hasse diagrams shown in Fig. 4.7 only the left one is a lattice. In the other two posets elements x, y have no join.

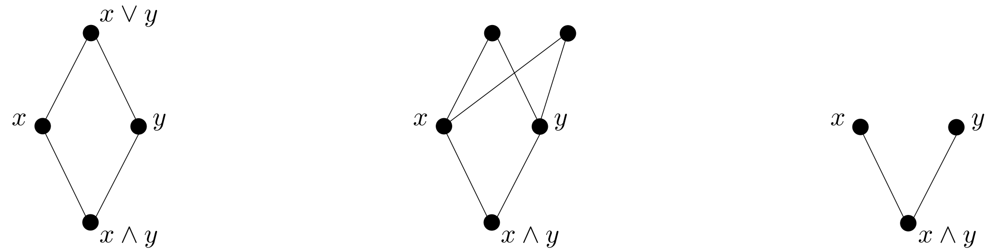


Figure 4.7: The poset on the left is a lattice, the other two are not.

Note that if $x \leq y$, then $x \vee y = y$ and $x \wedge y = x$.

Exercise. Every finite lattice has a unique *minimal* and a unique *maximal* elements (i.e., elements c and d such that $c \leq x \leq d$ for all x).

Lemma 4.22. *Boolean lattice B_n is indeed a lattice.*

Proof. Observe that if $X, Y \in B_n$, then $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$. □

Example 4.23. *Young lattice* $\mathbb{Y}$ is the poset of all Young diagrams ordered by inclusion. The covering relation in $\mathbb{Y}$ is defined as follows: $\lambda \lessdot \mu$ if μ has precisely one extra box.

Exercise. Check that union and intersection of Young diagrams is again a Young diagram. Show that $\mathbb{Y}$ is a lattice.

Example. *Partition lattice* Π_n consists of all set partitions of $[n]$ ordered by refinement: $\lambda \leq \mu$ if μ is obtained by joining together some blocks of λ. The covering relation is then following: $\lambda \lessdot \mu$ is μ is obtained by combining two blocks of λ.

Exercise. Show that Π_n is a lattice.

Definition 4.24. Let P be a poset. An *order ideal* of P is a set $I \subset P$ such that if $x \in I$ and $y \leq x$ then $y \in I$. Define an *order* on order ideals: $I \leq J$ if $I \subset J$. This define a *poset of order ideals* $J(P)$.

Example. A poset and its poset of order ideals are shown in Fig. 4.8.

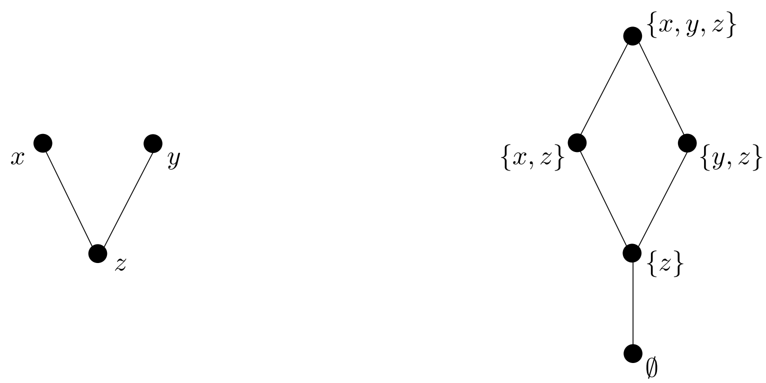


Figure 4.8: Hasse diagrams of a poset (left) and of its poset of order ideals (right).

Lemma 4.25. *$J(P)$ is a lattice.*

Proof. Since the order is defined as in the Boolean lattice, meet and join are the intersection and the union (which are also order ideals – check this!). □

Example 4.26.

- Consider a poset on $\mathbb{Z}_{\geq 0}$ with usual order. Then $J(\mathbb{Z}_{\geq 0}) \cong \mathbb{Z}_{\geq 0} \cup \{\infty\}$ (an order ideal corresponds to its maximal element).
- $J(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}) \cong \mathbb{Y}$ (Young lattice)
- We can also interpret B_n as $J([n])$, where all elements of $[n]$ are incomparable.

Definition 4.27. A Lattice is *distributive* if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \text{and} \qquad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Remark. An ordered field is not a distributive lattice: only one of the distributive laws holds.

Exercise. $J(P)$ is a distributive lattice.

Theorem 4.28 (Fundamental Theorem of Finite Distributive Lattices, Birkhoff)**.** *Any finite distributive lattice is a poset of order ideals for some finite poset.*

Proof. Let P be any poset. We call $z \in P$ *join-irreducible* if $z \neq x \vee y$ for any incomparable $x, y \in P$, and z is not a minimal element of P.

Now, let L be a finite distributive lattice. Define P to be the poset of all join-irreducible elements of L (with the order inherited from L). We will prove that $L = J(P)$.

Take $x \in L$, and consider $I_x = \{y \in P \mid y \leq x\} \subset P$. Clearly, $I_x \in J(P)$: if $y \in I_x$, $z \in P$ and $z \leq y$ then $z \leq y \leq x$ and thus $z \in I_x$.

Thus, we constructed a map from L to $J(P)$ taking $x \in L$ to $I_x \in J(P)$. We need to show that this map is injective and surjective. Injectivity follows from the following exercise:

Exercise. For every $x \in L$, $x = \vee\{t \mid t \in I_x\}$.

Indeed, the Exercise implies that if $I_x = I_y$ and $x \neq y$, then the same set have two different joins, which is impossible in a lattice. To surjectivity, we will prove the following claim.

Claim. Let $I \in J(P)$, take $x = \vee\{t \mid t \in I\}$. Then $I = I_x$.

The claim explicitly states that the map $x \mapsto I_x$ is surjective: for every order ideal of P we find a preimage. Therefore, we are left to prove the claim.

□

Proof of the claim. First, $I \subset I_x$: since $x = \vee\{t \mid t \in I\}$, $t \leq x$ for all $t \in I$, and thus every $t \in I$ also lies in I_x by the definition of I_x. We are left to prove that $I_x \subset I$.

Since $x = \vee\{y \mid y \in I_x\} = \vee\{t \mid t \in I\}$, we have

$$x = \vee\{t \mid t \in I\} = \vee\{y \mid y \in I_x\}$$

Take any $s \in I_x$, we want to show that $s \in I$. Take the meet $s \wedge x$, expanding both sides of the equality above by the distributive law. Then we get

$$\vee\{t \wedge s \mid t \in I\} = \vee\{y \wedge s \mid y \in I_x\}$$

In the RHS, every $y \wedge s \leq s$, and since $s \in I_x$ there is also a term $s \wedge s = s$, so the RHS is equal to s, and thus the LHS is equal to s. Since $s \in I_x$, s is join-irreducible, which means that at least one of the elements of the set $\{t \wedge s \mid t \in I\}$ must be equal to s. But $s \wedge t = s$ implies $s \leq t$, and since I is an order ideal, $s \leq t \in I$ implies $s \in I$.

□

Example. Let λ be a Young diagram. Define a lattice $\mathbb{Y}_\lambda$ consisting of all Young diagrams that fit inside λ (with the order inherited from $\mathbb{Y}$). The $\mathbb{Y}_\lambda$ is a distributive lattice (check this!). Join-irreducible elements of $\mathbb{Y}_\lambda$ are Young diagrams with precisely one corner, i.e. rectangles. Define P_λ as a poset on boxes of λ, with $(i, j) \leq (k, l)$ if $i \leq k$ and $j \leq l$. Then $J(P_\lambda) = \mathbb{Y}_\lambda$ (check this!).

4.5 Linear extensions of posets

Let P be a finite poset, $|P| = n$.

Definition 4.29. A *linear extension* of a poset P is a bijective map $f : P \to [n]$ such that $x \leq_P y$ implies $f(x) \leq f(y)$. Denote by $\mathrm{ext}\,(P)$ the number of linear extensions of P.

Example. Let $P = P_\lambda$, where $\lambda = (2,2) \vdash 4$. Then linear extensions can be identified with SYT of shape λ. In particular, $\mathrm{ext}\,(P) = 2$.

Lemma 4.30. *Let P be a finite poset, and let $J(P)$ be its poset of order ideals. Then $\mathrm{ext}\,(P)$ is equal to the number of maximal saturated chains in $J(P)$.*

Corollary 4.31. *The number of SYT of shape λ, f_λ, is equal to the number of saturated chains from $\emptyset$ to λ in $\mathbb{Y}_\lambda$.*

Proof of Lemma 4.30. We want to construct a bijection between extensions of P and maximal saturated chains in $J(P)$. Let $\varphi : P \to [n]$ be an extension, $j \in [n]$. Define an order ideal $I_j = \varphi^{-1}([j])$, then we get a saturated chain $\emptyset \lessdot I_1 \lessdot \ldots \lessdot I_n = P$.

Exercise. Show that φ is a bijection.

□

5 Robinson-Schensted correspondence

5.1 Algorithm

Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ be an integer partition. Recall that f_λ is the number of standard Young tableaux (SYT) of shape λ, which is also equal to the number of saturated chains in the Young lattice $\mathbb{Y}$ from $\emptyset$ to λ.

Theorem 5.1.

$$\sum_{\lambda \vdash n} (f_\lambda)^2 = n!$$

Example 5.2. For $n = 3$, there are precisely three Young diagrams. Two of these correspond to a unique SYT, and the other has two SYT. Then $1^2 + 2^2 + 1^2 = 6 = 3!$.

Remark (can be ignored by those not taking Representation Theory)**.** Theorem 5.1 can be interpreted as a fact from the representation theory of symmetric groups. Irreducible representations of S_n are parameterized by integer partitions of n, i.e. by Young diagrams. Then f_λ is precisely the dimension of the irreducible representation V_λ, $n!$ is the order of S_n, and thus the theorem says that sum of squares of dimensions of irreducible representations is equal to the order of the group (which is true for any finite group and can be proved by considering regular representation).

We will prove Thm 5.1 combinatorially by constructing a bijection between permutations in S_n and pairs (P, Q) of SYT of the same shape $\lambda \vdash n$.

The algorithm is due to Robinson–Schensted, with a generalization due to Knuth, and is usually referred as RSK (Robinson–Schensted–Knuth).

Given $w = w_1 w_2 \ldots w_n \in S_n$, we will construct an *insertion tableau* P and a *recording tableau* Q of the same shape step by step.

Example 5.3. Let $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$. In the beginning, $P = \emptyset$ and $Q = \emptyset$. We start adding w_i in turn. First, let us consider how P is changing. Every new w_i is inserted in the first row.

- At the first step, 3 is inserted in the box $(1,1)$.
- At the second step, 2 is inserted. It cannot be inserted in the box $(1,2)$ as $2 > 3$, so it is inserted in the box $(1,1)$ and thus pushes down 3 into a new box $(2,1)$.
- At the third step, 6 is inserted in the box $(1,2)$.
- At the fourth step, 1 is inserted in the box $(1,1)$, and thus pushes down 2 into the second row. 2, in its turn, pushes down 3 to the third row into a new box $(3,1)$.
- At the fifth step, 5 is inserted in the box $(1,2)$, and thus pushes down 6 into the second row, where 6 forms a new box $(2,2)$.
- At the sixth step, 7 is inserted in the box $(1,3)$.
- At the last step, 4 is inserted in the box $(1,2)$, and thus pushes down 5 into the second row. 5, in its turn, pushes down 6 to the third row into a new box $(3,2)$.

As a result, we get the following SYT of shape $\lambda = (3,2,2)$:

$$P = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 6 \\ \cline{1-2}\end{array}$$

Now, Q *records* the number of the step each box of P was introduced for the first time. For example, the box $(1,3)$ was introduced at the sixth step, so there will be 6 in it, and the box $(3,1)$ has shown up at the fourth step, so there will be 4 there. As a result, we have

$$Q = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array}$$

Let us formalize the procedure. The step (inserting w_i to the tableau P_{i-1} obtained after inserting w_{i-1}) consists of the following:

- if w_i is greater than all entries in the first row of P_{i-1} then w_i is added to the end of the first row;
- otherwise, take the minimal w_j in the first row which is greater than w_i, substitute w_j with w_i, and insert w_j in the tableau obtained by removing the first row from P_{i-1}; i.e., if w_j is greater than all entries in the second row of P_{i-1}, then w_j is added to the end of the second row; otherwise, find the minimal w_k in the second row which is greater than w_j, substitute w_k with w_j, and push w_k into the third row, etc.

Example. Show that P and Q are SYT.

Both P and Q are SYT of the same shape λ, which is called the *Schensted shape of w*.

To verify that the procedure works, we need to check that the map $w \mapsto (P,Q)$ is injective and surjective. For this, we can construct the inverse map.

Example 5.4. Let us take $P = P_7$ and $Q = Q_7$ we got in Example 5.3, and try to reconstruct $w \in S_7$. According to Q, the last box to appear was $(3, 2)$ (with 6 in it). It is in row 3, so 6 was pushed down by some element from row 2. The maximal number in row 2 which is less than 6 is 5, which implies that 6 was pushed down by 5, and 5 was pushed down to row 2 by someone from the first row. The maximal number in the first row which is less than 5 is 4, which implies that 5 was pushed down by 4, and thus 4 is precisely the number inserted at the last step.

Therefore, we have found $w_7 = 4$ and reconstructed P_6 (and, obviously, Q_6), where

$$P_6 = \begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad Q_6 = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

We now repeat this procedure for the box $(1, 3)$ of (P_6, Q_6) which, according to Q_6, is the last one to appear, etc. After seven steps we will reconstruct the permutation w.

5.2 Properties of RSK

We will now look at some corollaries of the algorithm, as well as at some ot its properties.

Theorem 5.5. *Let λ be the Schensted shape of $w \in S_n$, and let (P, Q) be the corresponding SYT. Then*

(1) *λ_1 is the maximal size of an increasing subsequence in w;*

(2) *λ'_1 (= the number of rows in λ) is the maximal size of a decreasing subsequence in w.*

Remark. Theorem 5.5 immediately implies Erdos-Szekeres Theorem.

Remark. The Schensted shape of $w \in S_n$ is precisely the Young diagram λ constructed by the poset P_w in the Greene's Theorem. See Appendix 1 (written by Sergey Fomin) to Stanley's "Enumerative Combinatorics, vol. II".

Exercise 5.6. Denote by $r_x(P)$ an insertion of x in a partial tableau P in the RSK algorithm. Suppose that during $r_x(P)$ the elements $x_1, \dots, x_k$ are pushed down from rows $1, 2, \dots, k$ and columns $j_1, j_2, \dots, j_k$ respectively. Then

(a) $x < x_1 < \dots < x_k$;

(b) $j_1 \geq \dots \geq j_k$;

(c) if $P' = r_x(P)$, then $P'_{i,j} \leq P_{i,j}$ for all i, j.

Statement (1) of Theorem 5.5 immediately follows from the following lemma.

Lemma 5.7. *Let $w = w_1 w_2 \dots w_n \in S_n$, denote by P_k a partial tableau obtained after insertion of $w_1, \dots, w_k$. Let w_k enter P_{k-1} in column j. Then the longest increasing subsequence ending in w_k has length j.*

Proof. We use induction on k. The base $(k = 1)$ is obvious.

Let us first prove the existence of an increasing subsequence of length j ending in w_k. Let w_i be the element of P_{k-1} in the box $(1, j-1)$. Since w_k is inserted in the box $(1, j)$, $w_i < w_k$. By induction, there exists an increasing subsequence ending in w_i of length $j - 1$. By adding w_k, we obtain an increasing subsequence ending in w_k of length j.

No, let us prove the maximality. Suppose there exists an increasing subsequence ending in w_k of length greater than j. Take this subsequence, and let w_i be the element preceding w_k, $i < k$. By the induction assumption, w_i is inserted in box $(1, m)$, $m \geq j$ (as there exists an increasing subsequence ending in w_i of length at least j). Therefore, for the element w_p in the box $(1, j)$ of P_i we have $w_p \leq w_i < w_k$. Let w_q be the element in the box $(1, j)$ of P_{k-1}. By Exercise 5.6(c), $w_q \leq w_p$ (as $k - 1 \geq i$), and thus $w_q < w_k$. However, during the insertion of w_k it pushes down w_q, which implies that $w_k < w_q$, so we came to a contradiction.

□

We are left to prove statement (2) of the theorem.

Recall that we denote by $r_x(P)$ a step of the RSK algorithm consisting of inserting x in a partial tableau P. Define $c_x(P)$ as a (purely formal) *row insertion* of x into P. This can be understood as transposing P, then doing $r_x(P^t)$, and then transposing the result again (here by the transpose P^t of P we mean the reflection of P with respect to the main diagonal).

Lemma 5.8. *Let P be a partial tableau, $x, y \notin P$. Then $c_y r_x(P) = r_x c_y(P)$.*

Proof. The proof is case-by-case, we will consider some and leave others as an exercise.

Assume first that $y > x$ and y is greater than all elements of P. Then c_y places y at the end of the first column, and thus $r_x c_y(P)$ is just $r_x(P)$ with additional y attached to the bottom of the first column. It is clear that $c_y r_x(P)$ does precisely the same.

If we assume that x is the maximal element, then the same proof works (just need to transpose the whole picture).

Exercise. Complete the proof of the lemma.

□

Lemma 5.9. *Let $w = w_1 w_2 \ldots w_n \in S_n$, and let P be the insertion tableau of w. Define $w^r = w_n w_{n-1} \ldots w_2 w_1$. Then the insertion tableau of w^r is P^t.*

Proof. We can write $P = P(w) = r_{w_n} r_{w_{n-1}} \ldots r_{w_2} r_{w_1}(\emptyset)$.

Following this, and by using Lemma 5.8, we have

$$P(w^r) = r_{w_1} r_{w_2} \ldots r_{w_n}(\emptyset) = r_{w_1} r_{w_2} \ldots r_{w_{n-1}} c_{w_n}(\emptyset) = c_{w_n} r_{w_1} r_{w_2} \ldots r_{w_{n-1}}(\emptyset)$$

We can now continue, with

$$P(w^r) = c_{w_n} r_{w_1} r_{w_2} \ldots r_{w_{n-1}}(\emptyset) = c_{w_n} r_{w_1} r_{w_2} \ldots c_{w_{n-1}}(\emptyset) = c_{w_n} c_{w_{n-1}} r_{w_1} r_{w_2} \ldots r_{w_{n-2}}(\emptyset)$$

Applying this transformation n times, we get

$$P(w^r) = r_{w_1} r_{w_2} \ldots r_{w_n}(\emptyset) = c_{w_n} c_{w_{n-1}} \ldots c_{w_2} c_{w_1}(\emptyset) = P^t$$

as required. □

We can now complete the proof of Theorem 5.5(2).

Consider w^r, its increasing subsequences are precisely decreasing subsequences of w. Therefore, the maximal length of a decreasing subsequence of w is equal to the maximal length of an increasing subsequence of w^r, which, by (1), is equal to $(\lambda')_1$, where λ' is the Schensted shape of w^r. By Lemma 5.9, λ' is conjugate to λ, so $(\lambda)'_1$ is precisely the size of the first column of λ.

We will now explore more symmetries of RSK.

Example 5.10. Recall from Example 5.3 that for $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$ we have

$$P = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 6 \\ \cline{1-2}\end{array} \qquad Q = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 & 7 \\ \cline{1-2}\end{array}$$

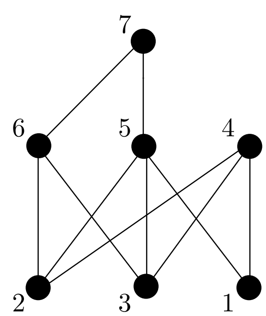
Let now take the insertion tableau $P' = Q$ and the recording tableau $Q' = P$, which permutation does this pair correspond to? An application of the RSK algorithm leads to $w' = 4\,2\,1\,7\,5\,3\,6 \in S_7$. Now, observe that $w = (1\,3\,6\,7\,4)(2)(5)$ and $w' = (1\,4\,7\,6\,3)(2)(5)$, so $w' = w^{-1}$.

Theorem 5.11. *Let the application of the RSK takes w to (P, Q). Then w^{-1} is taken to (Q, P).*

Recall that, given $w \in S_n$, one can define a poset P_w on $[n]$ with order $w_i <_{P_w} w_j$ if $w_i < w_j$ and $i < j$. Then chains of P_w are increasing subsequences of w, and antichains are decreasing subsequences of w.

Denote by P_1 the set of minimal elements of P_w, by P_2 the set of minimal elements of $P_w \setminus P_1$, and then by P_i the set of minimal elements of $P_w \setminus \bigcup_{j<i} P_j$. Note that every P_i is an antichain of P_w.

Example. Let $w = 3\,2\,6\,1\,5\,7\,4 \in S_7$, we have already seen that the Hasse diagram of P_w is



Then $P_1 = \{1, 2, 3\}$, $P_2 = \{4, 5, 6\}$, and $P_3 = \{7\}$ (cf columns of the the insertion tableau).

Write w in 2-line notation: $w = \begin{matrix} 1\,2\,3\,4\,5\,6\,7 \\ 3\,2\,6\,1\,5\,7\,4 \end{matrix}$. Then we can interpret elements of P_w as $\begin{matrix} i \\ w_i \end{matrix}$. In particular, we can write every P_i ordering its elements by the top number:

$$\begin{aligned}
P_1 &= \left\{ \begin{matrix}1\\3\end{matrix}, \begin{matrix}2\\2\end{matrix}, \begin{matrix}4\\1\end{matrix} \right\} &&= \left\{ \begin{matrix}u_{11}\\w_{11}\end{matrix}, \begin{matrix}u_{12}\\w_{12}\end{matrix}, \begin{matrix}u_{13}\\w_{13}\end{matrix} \right\} \\
P_2 &= \left\{ \begin{matrix}3\\6\end{matrix}, \begin{matrix}5\\5\end{matrix}, \begin{matrix}7\\4\end{matrix} \right\} &&= \left\{ \begin{matrix}u_{21}\\w_{21}\end{matrix}, \begin{matrix}u_{22}\\w_{22}\end{matrix}, \begin{matrix}u_{23}\\w_{23}\end{matrix} \right\} \\
P_3 &= \left\{ \begin{matrix}6\\7\end{matrix} \right\} &&= \left\{ \begin{matrix}u_{31}\\w_{31}\end{matrix} \right\}
\end{aligned}$$

where $\begin{matrix} u_{ij} \\ w_{ij} \end{matrix}$ denotes j-th element of P_i.

Note that the first row of the insertion tableau P is $1\,4\,7 = w_{13}\,w_{23}\,w_{31}$, and the first row of the recording tableau Q is $1\,3\,6 = u_{11}\,u_{21}\,u_{31}$.

Remark. In general, if $P_j = \left\{ \begin{matrix}u_{j1}\\w_{j1}\end{matrix}, \begin{matrix}u_{j2}\\w_{j2}\end{matrix}, \dots, \begin{matrix}u_{jn_j}\\w_{jn_j}\end{matrix} \right\}$ (where we always assume that $u_{j1} < u_{j2} < \cdots < u_{jn_j}$), then $w_{j1} > w_{j2} > \cdots > w_{jn_j}$ as all these elements compose an antichain of P_w.

Remark. While applying RSK to a permutation $w = \begin{matrix} 1 & 2 & \dots & n \\ w_1 & w_2 & \dots & w_n \end{matrix}$ we insert the elements of the second row into tableau P and the elements of the first row into tableau Q. Note that the algorithm is completely defined by the relations $w_i < w_j$ for every pair (i, j). Thus, if we substitute the first row by any increasing sequence, and the second row by any n distinct numbers with the same relations, then we will get tableaux P' and Q' of the same shape as of P and Q (these are not SYT anymore, although the numbers will still be increasing to the right and to the bottom). Moreover, we can apply RSK to any two-row array of numbers (such that numbers in every row are distinct): sort the columns so that the top row is increasing, and then apply the procedure described above. We will use this modification of RSK to complete the proof of the theorem.

Lemma 5.12. *The first row of the insertion tableau P is $w_{1n_1}\, w_{2n_2}\, w_{3n_3} \dots$, and the first row of the recording tableau Q is $u_{11}\, u_{21}\, u_{31} \dots$. In other words, the first rows of P and Q consist of minimal w and u from every P_j.*

Proof. Let $w = w_1\, w_2\, \dots\, w_n \in S_n$, we use induction on n. If $n = 1$ then the statement is trivial.

Denote $\widetilde{w} = w_1\, w_2\, \dots\, w_{n-1}$, and let $\widetilde{P}$ and $\widetilde{Q}$ be the corresponding insertion and recording tableaux (see the remark above: here the first row of $\widetilde{w}$ is still $[n-1]$, but the second row may miss any one element of $[n]$). Let $\widetilde{P}_j$ be the corresponding antichains of $P_{\widetilde{w}}$, $\widetilde{P}_j = \left\{ \begin{matrix} \widetilde{u}_{j1} \\ \widetilde{w}_{j1} \end{matrix}, \begin{matrix} \widetilde{u}_{j2} \\ \widetilde{w}_{j2} \end{matrix}, \dots, \begin{matrix} \widetilde{u}_{jm_j} \\ \widetilde{w}_{jm_j} \end{matrix} \right\}$, where $j = 1, \dots, l$. By the induction assumption, the first row of $\widetilde{P}$ is $\widetilde{w}_{1m_1}\, \widetilde{w}_{2m_2}\, \dots\, \widetilde{w}_{lm_l}$, and the first row of $\widetilde{Q}$ is $\widetilde{u}_{11}\, \widetilde{u}_{21}\, \dots\, \widetilde{u}_{l1}$.

Insert w_n in $\widetilde{P}$. If $w_n > \widetilde{w}_{jm_j}$ for all j, then we insert w_n in the $(l+1)$-st column. At the same time, this means that $\begin{matrix} n \\ w_n \end{matrix}$ cannot be added to any of l antichains P_j, so it forms a new antichain P_{l+1} in P_w, and thus we can write $\begin{matrix} n \\ w_n \end{matrix} = \begin{matrix} u_{l+1\,1} \\ w_{l+1\,1} \end{matrix} = \begin{matrix} u_{l+1\,m_{l+1}} \\ w_{l+1\,m_{l+1}} \end{matrix}$, so the first row of P is $w_{1m_1}\, w_{2m_2}\, \dots\, w_{lm_l}\, w_{l+1\,m_{l+1}}$, and the first row of Q is $u_{11}\, u_{21}\, \dots\, u_{l1}, u_{l+1\,1}$, as desired.

Assume now that $w_n < \widetilde{w}_{jm_j}$ for some j. Then $\widetilde{P}_j \cup \left\{ \begin{matrix} n \\ w_n \end{matrix} \right\}$ is an antichain of P_w (note that there may be many such j). One can easily see that $\begin{matrix} n \\ w_n \end{matrix}$ belongs to P_j with minimal j amongst those with $w_n < \widetilde{w}_{jm_j}$. For example, if $w_n < \widetilde{w}_{1m_1}$, then $\begin{matrix} n \\ w_n \end{matrix}$ is incomparable with all minimal elements of P_w, and thus it is a minimal element itself. Therefore, the element $\begin{matrix} n \\ w_n \end{matrix}$ is inserted in column j if and only if $\begin{matrix} n \\ w_n \end{matrix} = \begin{matrix} u_{jm_j} \\ w_{jm_j} \end{matrix}$, so again the first row of P is of the form required in the lemma. Since we do not start the new column, Q stays intact, so it also has the required form.

$\square$

Exercise 5.13. The poset P_w is isomorphic to the poset $P_{w^{-1}}$, with the isomorphism given by $\begin{matrix} i \\ w_i \end{matrix} \mapsto \begin{matrix} w_i \\ i \end{matrix}$.

Proof of Theorem 5.11. Let the insertion and recording tableaux of w^{-1} be P' and Q'. Denote by P'_j the corresponding antichains of the poset $P_{w^{-1}}$. According to Exercise 5.13, $P'_j = \left\{ \begin{matrix} w_{j1} \\ u_{j1} \end{matrix}, \begin{matrix} w_{j2} \\ u_{j2} \end{matrix}, \dots, \begin{matrix} w_{jn_j} \\ u_{jn_j} \end{matrix} \right\}$, so the minimal elements in the second row are u_{j1}, and the minimal elements in the first row are w_{jn_j}. According to Lemma 5.12, the first row of P' is now $u_{11}\, u_{21}\, u_{31} \dots$, and the first row of Q' is $w_{1n_1}\, w_{2n_2}\, w_{3n_3} \dots$, which are precisely first rows of Q and P respectively.

Now the plan is the following: denote by $\bar{P}$ and $\bar{Q}$ the tableaux P and Q with their first rows removed, and find a two-rows array $\bar{w}$ (i.e. a "permutation") which results in $(\bar{P}, \bar{Q})$ under RSK. We already know the set of elements in both rows, the question is how to match them.

Consider two elements $\begin{matrix} u_{jq} \\ w_{jq} \end{matrix}$, $q < m_j$, and $\begin{matrix} u_{rs} \\ w_{rs} \end{matrix}$, $s < m_r$, then both w_{jq} and w_{rs} are in $\bar{P}$. Thus, there are two elements which pushed these two down.

Exercise. $\begin{matrix} u_{jq} \\ w_{jq} \end{matrix}$ is pushed down from the first row of P by $\begin{matrix} u_{j\,q+1} \\ w_{j\,q+1} \end{matrix}$.

Therefore, w_{jq} enters $\bar{P}$ before w_{rs} if and only if $u_{j\,q+1} < u_{r\,s+1}$. This implies that the following array $\bar{w}$ is taken to $(\bar{P}, \bar{Q})$ (check this!):

$$\bar{w} = \begin{matrix} u_{12} \dots & u_{1m_1} & u_{22} \dots & u_{2m_2} & \dots & u_{l2} \dots & u_{lm_l} \\ w_{11} \dots & w_{1\,m_1-1} & w_{21} \dots & w_{2\,m_2-1} & \dots & w_{l1} \dots & w_{l\,m_l-1} \end{matrix}$$

for some l, where the columns should be permuted for the first row to be increasing.

Now, performing a similar exercise for w^{-1}, we see that

$$\overline{w^{-1}} = \begin{matrix} w_{1\,m_1-1} \dots & w_{11} & w_{2\,m_2-1} \dots & w_{21} & \dots & w_{l\,m_l-1} \dots & w_{l1} \\ u_{1m_1} & \dots u_{12} & u_{2m_2} & \dots u_{22} & \dots & u_{lm_l} & \dots u_{l2} \end{matrix}.$$

Observe that $\overline{w^{-1}} = (\bar{w})^{-1}$, and thus, by the induction assumption, $\bar{P}' = \bar{Q}$ and $\bar{Q}' = \bar{P}$. As we have already proved that the first rows also coincide, this implies that $P' = Q$ and $Q' = P$.

□

6 Games on graphs

Let $G = (V, E)$ be a graph with the set of vertices V (where $|V| = n$ and the vertices are identified with elements of $[n]$), and the set of edges E, where $E \subset V \times V$ (we remove the diagonal from $V \times V$ and identify (i, j) with (j, i)).

Denote by $N(i)$ the set of *neighbors* of i, i.e. vertices connected to i by an edge. A *configuration* is a non-negative integer vector $\boldsymbol{c} = (c_1, \dots, c_n)$ – this can be understood as we put c_i chips in a vertex i.

6.1 Reflection game

We call a vertex i *unstable* if $2c_i < \sum_{j \in N(i)} c_j$, and *stable* otherwise.

The *initial configuration* is a vector with all $c_j = 0$ except for a single $c_i = 1$.

A *move* consists of choosing any unstable vertex i and changing the configuration as follows: $c_i \mapsto -c_i + \sum_{j \in N(i)} c_j$, while all other c_j stay intact. The *goal* of the game is to make every vertex stable.

Example. For G being a path with three vertices, there are three possible initial configurations, they all lead to a final configuration $(1, 1, 1)$.

We can ask several questions:

- For which graphs the game can stop?
- For which graphs the game stops for any sequence of moves?
- What are possible final configurations?

Example. For G being a cycle of length three, the game never stops.

Exercise.
- A *subgraph* of $G = (V, E)$ consists of a subset V' of V and all edges from E joining elements of V'. Show that if the game never stops on some subgraph of G, then it never stops on G.

- Show that if G has a cycle, then the game never stops.
- The *valence* of a vertex of G is the number of edges incident to it. Show that if G has a vertex of valence at least four, then the game never stops.
- Show that if G has at least two vertices of valence three, then the game never stops.
- Let G be a tree with a unique vertex v_0 of valence three. Denote the lengths of the "legs" (including v_0) by p, q, r, see Fig. 6.1. Show that if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$ then the game never stops.

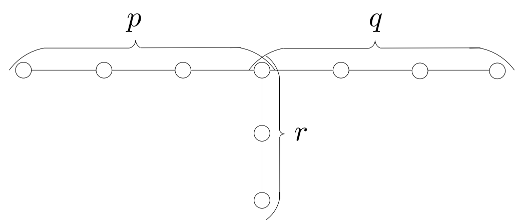


Figure 6.1: Tree with precisely one vertex of valence at least three

The remaining graphs are shown in Fig. 6.2.

Exercise. Show that for the graphs in Fig. 6.2 any sequence of moves terminates, and the final configuration is always the same (such graphs are called *graphs of finite type*).

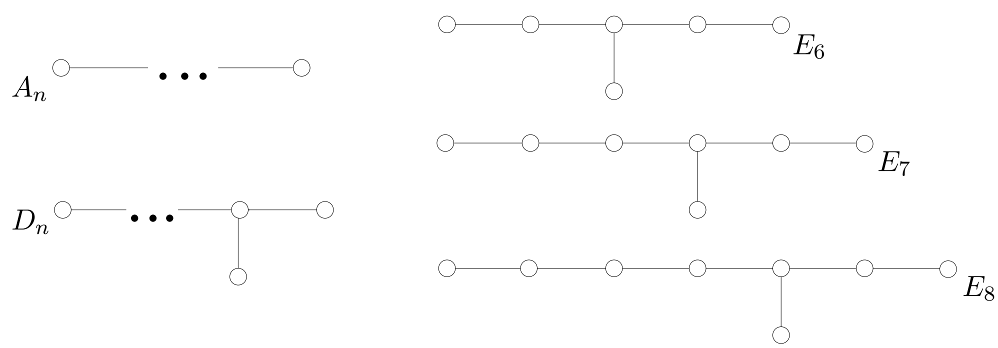


Figure 6.2: Graphs of finite type

6.2 Cartan firing

We now call a vertex i *unstable* if $c_i > 1$, and *stable* otherwise.

We can start with any initial configuration.

A *move* consists of choosing any unstable vertex i and changing the configuration as follows: $c_i \mapsto c_i - 2$, $c_j \mapsto c_j + 1$ if $j \in N(i)$, and c_j stays intact otherwise. The goal of the game is to make every vertex stable.

Example. For G being a path with three vertices and the initial configuration $(2, 3, 2)$, there is a sequence of moves which terminates.

Given a graph G, define its *Cartan matrix* $A_G = (a_{ij})$ as follows: $a_{ii} = 2$, $a_{ij} = -1$ if $i \in N(j)$, and $a_{ij} = 0$ otherwise. Then the move at vertex i can be understood as $\boldsymbol{c} \mapsto \boldsymbol{c} - A_i$, where A_i is the i-th row of A_G.

Theorem 6.1. *Let G be a graph, and let $A = A_G$ be its Cartan matrix. Then the following are equivalent:*

(1) For any initial configuration and any sequence of moves the game stops.

(2) There exists a positive vector $v = (v_1, \ldots, v_n)$, $v_i > 0$, such that Av has also all coordinates positive.

(3) A is positive definite.

7 Reflection groups

7.1 Linear reflections and reflection groups

We consider $\mathbb{R}^n$ with standard Euclidean scalar product $(\cdot,\cdot)$ (i.e., dot product).

Definition 7.1. Let $\alpha \in \mathbb{R}^n$. A *reflection* with respect to α is a linear map $r_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$r_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha$$

for $v \in \mathbb{R}^n$.

It is easy to see that r_α is characterized by the following two properties: it takes α to $-\alpha$ and preserves pointwise the orthogonal complement $\alpha^\perp$.

Exercise 7.2. $r_\alpha \in O_n(\mathbb{R})$, i.e. $(r_\alpha(u), r_\alpha(v)) = (u,v)$; also, r_α is an involution, i.e. $r_\alpha^2 = \mathrm{id}$.

Definition 7.3. A *reflection group* is a group generated by reflections.

Remark. Usually "reflection group" means a *discrete reflection group*, which requires some additional geometrical properties to hold. We will mainly be interested in *finite* reflection groups, and for these there are no extra requirements.

Example 7.4.
- Consider vectors $\alpha = (1,0)$ and $\beta = (\cos\frac{(m-1)\pi}{m}, \sin\frac{(m-1)\pi}{m})$. Then $r_\alpha r_\beta$ is a rotation by $2\pi/m$, and the group generated by r_α and r_β is a dihedral group of order $2m$ (denoted by $I_2(m)$).
- The symmetric group S_{n+1} acts on $\mathbb{R}^{n+1}$ by permutation of coordinates, preserving the hyperplane $V = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i = 0\}$. Every transposition (ij) is a reflection in a plane $x_i - x_j = 0$, i.e. with respect to the vector $e_i - e_j$. As S_{n+1} is generated by transpositions, it is a reflection group in V.

Definition 7.5. For reflection r_α, $\alpha \in \mathbb{R}^n$, the orthogonal complement $\alpha^\perp$ is called a *mirror* of r_α.

Lemma 7.6. *Let $g \in O_n(\mathbb{R})$, $\alpha \in \mathbb{R}^n$. Then $gr_\alpha g^{-1} = r_{g\alpha}$.*

Proof. We need to prove that $gr_\alpha g^{-1}$ fixes every point of $\langle g\alpha\rangle^\perp$ and takes $g\alpha$ to $-g\alpha$.

Let $(v, g\alpha) = 0$. Since $g \in O_n(\mathbb{R})$, this implies that $(g^{-1}v, \alpha) = 0$. Then

$$gr_\alpha g^{-1}(v) = g(r_\alpha(g^{-1}(v)) = g((g^{-1}(v)) = v,$$

so $gr_\alpha g^{-1}$ preserves $\langle g\alpha\rangle^\perp$ pointwise.

Also, $gr_\alpha g^{-1}(g\alpha) = gr_\alpha(\alpha) = g(-\alpha) = -g\alpha$, as required. □

In general, what can we say about *finite* reflection groups in $\mathbb{R}^n$?

First, since every reflection is orthogonal, any reflection group is a subgroup of $O_n(\mathbb{R})$. Given a finite reflection group G in $\mathbb{R}^n$, Lemma 7.6 implies that the set of mirrors of reflections of G is invariant under the action of G (i.e., G permutes its mirrors).

The set of mirrors of G decomposes $\mathbb{R}^n$ into polyhedral cones – we call them *chambers*, and the mirrors bounding a chamber are called its *walls*.

Remark. Note that, due to the invariance of the set of mirrors under G, any two chambers sharing a wall a taken to each other by the reflection in the common wall. Indeed, if we take chambers C_1 and C_2 sharing a wall $\alpha^\perp$, we can consider $C' = r_\alpha C_1$. If C' is not a chamber, then there exists a mirror $\beta^\perp$ of reflection in G intersecting C'. Applying r_α to $\beta^\perp$, we see that the image intersects C_1, which contradicts C_1 being a chamber. Now, both C' and C_2 are chambers, and they clearly have a non-empty intersection, so they must coincide.

Recall that an action of a group on a set is *transitive* if the set consists of one orbit.

Theorem 7.7. *Let G be a finite reflection group in $\mathbb{R}^n$. Consider all mirrors of reflections of G, and take any connected component of the complement, call this chamber C_0. Denote $r_{\alpha_1}, \dots, r_{\alpha_k}$ the reflections with respect to the walls of C_0. Then*

(1) G is generated by $r_{\alpha_1}, \dots, r_{\alpha_k}$.

(2) G acts transitively on the set of chambers.

(3) The dihedral angles between walls of C_0 are of the type π/m_{ij}, $m_{ij} \in \mathbb{N}_{\geq 2}$.

(4) If $g \in G$ and $gC_0 = C_0$ then $g = \mathrm{id}$.

(5) G has presentation $G = \langle r_{\alpha_1}, \dots, r_{\alpha_k} \mid r_{\alpha_i}^2, (r_{\alpha_i} r_{\alpha_j})^{m_{ij}} \rangle$ (i.e. any relation on the generators follows from these two types of relations).

Proof. Denote r_{α_i} by s_i. Take any chamber C, connect it to C_0 by a path (which does not pass through an intersection of three or more chambers). Write down the sequence of chambers intersected by the path: $C_0, C_1, \dots, C_m = C$ (chambers may repeat in the sequence). Note that any two neighboring chambers in the sequence share a wall.

Since C_1 and C_0 share a wall (say, mirror of s_{i_1}), we can write $C_1 = s_{i_1} C_0$. By Lemma 7.6, walls of C_1 are precisely mirrors of reflections $s_{i_1} s_j s_{i_1}$, $j = 1, \dots, m$. Since C_2 and C_1 share a wall (say, mirror of $s_{i_1} s_{i_2} s_{i_1}$), we can write

$$C_2 = s_{i_1} s_{i_2} s_{i_1} C_1 = s_{i_1} s_{i_2} s_{i_1} (s_{i_1} C_0) = s_{i_1} s_{i_2} C_0.$$

Continuing along the path, we see that $C = C_m = s_{i_1} s_{i_2} \dots s_{i_m} C_0$, where s_{i_j} are reflections in the walls of C_0. This proves (2).

Moreover, we have proved that any reflection in G is conjugated to at least one of s_i, which proves (1).

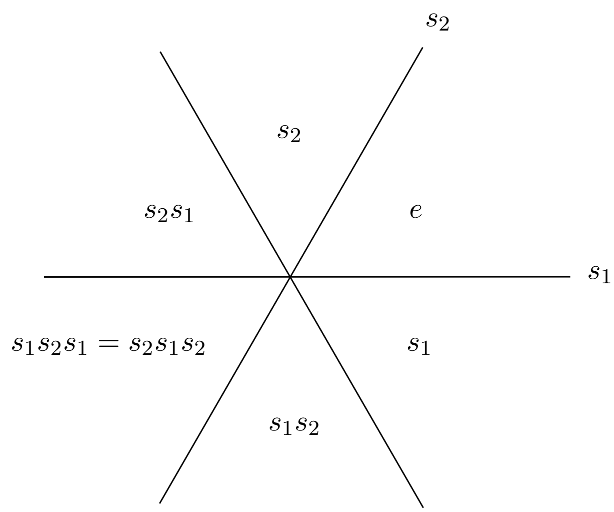
Now, let us prove (3). Take any s_i and S_j and consider the group generated by them. If the angle is not $p\pi/m$, then the order of $s_i s_j$ is infinite, which contradicts finiteness of G. Further, if $p \neq 1$ (we may assume that p and m are coprime), then the walls are separated by another mirror (this is actually a question about dihedral groups), which implies that there exists a mirror of G intersecting the interior of C_0 in contradiction with its definition, so (3) is also proved.

To prove (4) and (5) consider any word $s_{i_1} s_{i_2} \dots s_{i_m}$ realizing a path from C_0 to C_0 going through chambers $s_{i_1} s_{i_2} \dots s_{i_k} C_0$ for $k = 1, \dots, m-1$. Note that the relations in (5) do hold (as they hold in the corresponding dihedral groups). Moreover, they imply the same relations for reflections in walls of any chamber (due to Lemma 7.6). Further, using the relations we can contract the path to an empty one. More precisely, if a path intersects any single wall twice in a row, then applying the relation r^2 for r being the reflection in the corresponding wall we shorten the path; to go around an intersection of more than two chambers (which is a vector space of codimension two), one can use the relations (5) to substitute a subword of type $s_i s_j s_i \dots$ of length l by a word of type $s_j s_i s_j \dots$ of length $2m_{ij} - l$. Therefore, every path from C_0 to C_0 corresponds to a trivial element of G, and the word can be reduced to e by the required relations, which proves both (4) and (5).

□

Corollary 7.8. *Chambers of G are indexed by elements of G.*

Example. Consider $I_2(3) = S_3$, it has presentation $\langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$.



Definition 7.9. Let a group G act on an open connected set X. An open connected domain $C \subset X$ is called a *fundamental domain* of the action if the following conditions are satisfied:

- $X = \bigcup\limits_{g \in G} \overline{gC}$, where $\overline{gC}$ denotes the closure of gC;
- for any nontrivial $g \in G$, $C \cap gC = \emptyset$;
- there are finitely many $g \in G$ such that $\overline{C} \cap \overline{gC} \neq \emptyset$.

Corollary 7.10. *Any chamber C of a finite reflection group G is a fundamental domain of the action of G on $\mathbb{R}^n$. In particular, chambers are also called* fundamental chambers.

7.2 Classification of finite reflection groups

Theorem 7.7(3) has the following elementary corollary.

Corollary 7.11. *Let C be a chamber of a finite reflection group, and let r_α and r_β be two generating reflections, where α and β are outward normals to walls of C. Then $(\alpha, \beta) \leq 0$. In other words, all angles of C are acute (or non-obtuse, depending on the agreement whether $\pi/2$ is acute or not).*

Definition 7.12. A system of vectors is *indecomposable* if it cannot be split into two subsets orthogonal to each other.

Lemma 7.13. *Let $\{e_i\}$ be a finite indecomposable system of vectors in $\mathbb{R}^n$ such that $(e_i, e_j) \leq 0$ for all $i \neq j$. Then either all e_i are linearly independent, or there exists a unique (up to scaling) linear dependence, and all its coefficients are positive.*

Proof. Assume $\{e_i\}$ are linearly dependent, and there is a linear dependence with some coefficients positive and some non-positive. Define index sets I and J so that coefficients of $e_i > 0$ are positive if $i \in I$ and non-positive if $j \in J$. We then can write

$$\sum_{i \in I} c_i e_i = \sum_{j \in J} c_j e_j,$$

where $c_i > 0$ and $c_j \geq 0$. Denote $\alpha = \sum\limits_{i \in I} c_i e_i$ and $\beta = \sum\limits_{j \in J} c_j e_j$. Then

$$(\alpha, \beta) = \sum_{i \in I, j \in J} c_i c_j (e_i, e_j).$$

Since $\alpha = \beta$, the value above is non-negative. At the same time, all c_i and c_j are non-negative, and all (e_i, e_j) are non-positive, so the product is non-positive. Therefore, we conclude that $(\alpha, \beta) = 0$, and thus $\alpha = \beta = 0$.

Take any $j \in J$, then $(\alpha, e_j) = (0, e_j) = 0$. At the same time, $0 = (\alpha, e_j) = (\sum_{i \in I} c_i e_i, e_j) = \sum_{i \in I} c_i (e_i, e_j)$. Since all $c_i > 0$, this implies that $(e_i, e_j) = 0$ for all $i \in I$. As this holds for every $j \in J$, we get a contradiction with indecomposability of $\{e_i\}$.

Now, assume that there are two positive linear dependencies $\sum c_i e_i = 0 = \sum a_i e_i$. Since all a_i and c_i are positive, we can scale them such that $a_1 = c_1$. If the dependencies are still distinct, then subtracting one dependence from another we get a new dependence with the coefficient before e_i vanishing, which contradicts the statement we already proved.

□

Corollary 7.14. *If $\{e_i\}$ is a finite indecomposable system of vectors in $\mathbb{R}^n$ with $(e_i, e_j) \leq 0$ for $i \neq j$, then $\#\{e_i\} \leq n + 1$.*

Proof. Indeed, if there are $n + 2$ vectors, then there exists a linear dependence on any $n + 1$ of them, so there is a dependence with some coefficients vanishing, which contradicts Lemma 7.13. □

The next statement follows from the construction of the chambers.

Lemma 7.15. *Let C_0 be a chamber, and let α_i be outward normals to the walls of C_0. Then $C_0 = \{v \in \mathbb{R}^n \mid (v, \alpha_i) < 0\}$.*

Corollary 7.16. *In the assumptions of Lemma 7.15, $\#\{\alpha_i\} \leq n$. If G is* irreducible *(i.e., it has no invariant subspaces), then $\#\{\alpha_i\} = n$, and thus any chamber is a simplicial cone (i.e., any k walls intersect along an $(n - k)$-dimensional subspace).*

Proof. By Cor. 7.14, $\#\{\alpha_i\} \leq n + 1$. Assume there are $n + 1$ vectors, then there is a linear dependence $\sum c_i \alpha_i = 0$ with all $c_i > 0$. Thus, we can write

$$\alpha_{n+1} = -\sum_{i=1}^{n} \frac{c_i}{c_{n+1}} \alpha_i$$

Take any $v \in C_0$, we know that $(v, \alpha_i) < 0$ for every i. Then

$$(v, \alpha_{n+1}) = -(v, \sum_{i=1}^{n} \frac{c_i}{c_{n+1}} \alpha_i) = -\sum_{i=1}^{n} \frac{c_i}{c_{n+1}} (v, \alpha_i) > 0,$$

so we get a contradiction.

If $\#\{\alpha_i\} < n$, then the subspace $\{\alpha_i\}^{\perp}$ has positive dimension (and thus is non-empty) and is clearly invariant. □

In particular, we see that if G has no invariant subspaces, then the outward normals to every chamber compose a basis of $\mathbb{R}^n$.

Now, recall a simple fact from linear algebra. Let $\{e_i\}$ be a standard basis of $\mathbb{R}^n$, and let $\{\alpha_i\}$ be another basis. Denote by A the transformation matrix taking $\{e_i\}$ to $\{\alpha_i\}$ (i.e. columns of A are coordinates of $\{\alpha_i\}$ in terms of $\{e_i\}$). Then $(A^t A)_{ij} = (\alpha_i, \alpha_j)$. In particular, the matrix (α_i, α_j) (such matrix is called the *Gram matrix* of system of vectors $\{\alpha_i\}$) is the matrix of the dot-product in a different basis, and thus it is positive definite.

Therefore, to classify all finite reflection groups, we need (first) to find all positive-definite symmetric $n \times n$ matrices (a_{ij}) with $a_{ii} = 1$ and $a_{ij} = -\cos \frac{\pi}{m_{ij}}$ (we normalize outward normals to chambers to have unit length), and then see which of these matrices correspond to reflection groups.

Definition 7.17. Let G be a finite reflection group in $\mathbb{R}^n$, let $\{\alpha_i\}$ be outward unit normals to a chosen chamber. A *Coxeter diagram* of G is a (multi)graph defined as follows:

- vertices are indexed by α_i (or simply by natural numbers $1, \dots, n$);
- vertices i and j are joined by $m_{ij} - 2$ edges (or a simple edge with weight m_{ij}), where $(\alpha_i, \alpha_j) = -\cos \frac{\pi}{m_{ij}}$.

Example. Consider $I_2(3) = S_3 = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$. The product of two generators has order 3, the only m_{ij} is $m_{12} = 3$. As $3 - 2 = 1$, the Coxeter diagram consists of two vertices joined by one simple edge.

In general, the Coxeter diagram of $I_2(m)$ consists of two vertices and $m - 2$ edges between them (or one edge with weight m written above the edge).

Remark 7.18. A Coxeter diagram of G is connected if and only if the collection of vectors $\{\alpha_i\}$ is indecomposable, and if and only if G is irreducible.

Lemma 7.19. *A Coxeter diagram of a finite irreducible reflection group G in $\mathbb{R}^n$ (where $n > 2$) does not contain*

(1) cycles;

(2) edges of multiplicity at least 4;

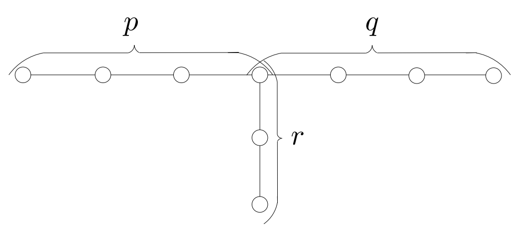
(3) two multiple edges;

(4) vertices of valence at least 4 (where by valence *we mean here the number of neighbors – this convention is not standard);*

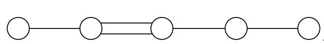
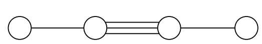
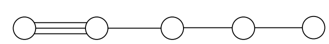
(5) both a vertex of valence 3 and a multiple edge;

(6) two vertices of valence at least 3;

(7) subdiagrams of type



with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$

(8) subdiagrams ,  *and* 

Proof. Every Coxeter diagram corresponds to a positive definite symmetric matrix $A = (a_{ij}) = (\alpha_i, \alpha_j)$. The plan of the proof is the following: for every prohibited subdiagram we either find a principal submatrix of A with non-positive determinant, or find a linear combination of α_i which has non-positive square.

(1) Let vertices $1, \dots, k$ form a chordless cycle (i.e. this is a minimal cycle we can find with respect to inclusion), consider the vector $v = \alpha_1 + \dots + \alpha_k$. Then

$$v^2 = (v, v) = (\sum \alpha_i, \sum \alpha_i) = \sum(\alpha_i, \alpha_i) + 2\sum_{i \neq j}(\alpha_i, \alpha_j) =$$

$$= k + 2 \sum_{i \leq k-1} (\alpha_i, \alpha_{i+1}) + 2(\alpha_k, \alpha_1) \leq k - 2k\frac{1}{2} = 0$$

as $(\alpha_i, \alpha_{i+1}) = -\|\alpha_i\|\|\alpha_{i+1}\| \cos \frac{\pi}{m_{i\,i+1}} = -\cos \frac{\pi}{m_{i\,i+1}} \leq -\frac{1}{2}$ for $m_{i\,i+1} \geq 3$.

(2) Suppose there is an edge of multiplicity at least 4 connecting vertices i and j. Since the diagram is connected and $n \geq 3$, there is a vertex k connected to one of i and j (say, to j). Since there are no cycles, k is not connected to i. Thus, we get a principal submatrix of A of the form

$$B(m_{ij}, m_{jk}) = \begin{pmatrix} 1 & -\cos\frac{\pi}{m_{ij}} & 0 \\ -\cos\frac{\pi}{m_{ij}} & 1 & -\cos\frac{\pi}{m_{jk}} \\ 0 & -\cos\frac{\pi}{m_{jk}} & 1 \end{pmatrix}.$$

Its determinant is equal to

$$\det B(m_{ij}, m_{jk}) = 1 - \cos^2\frac{\pi}{m_{ij}} - \cos^2\frac{\pi}{m_{jk}} \leq 1 - \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 = 1 - \frac{1}{4} - \frac{3}{4} = 0$$

as $\cos\frac{\pi}{m} \geq \frac{\sqrt{3}}{2}$ for $m \geq 6$.

(3) The determinant of matrix $B(4,4)$ is equal to zero, and $\det B(m_{ij}, m_{jk})$ is a decreasing function of both arguments. Thus, two multiple edges cannot have a common vertex.

Suppose that there are two multiple edges in a Coxeter diagram, we may assume that they are connected by a sequence of simple edges. Assume the multiple edges connect vertices 1 with 2, and k with $k+1$, and there are simple edges between vertices i and $i+1$ for $i = 2, \dots, k-1$. Consider vector $v = \sum_{i=2}^{k} \alpha_i$. It is easy to see that $(v,v) = 1$, $(v, \alpha_1) = (\alpha_2, \alpha_1)$, and $(v, \alpha_{k+1}) = (\alpha_k, \alpha_{k+1})$. Then the Gram matrix of vectors $\alpha_1, v, \alpha_{k+1}$ is precisely $B(m_{12}, m_{k\,k+1})$, so it has a non-positive determinant.

(4) Let 1 be a vertex with at least four neighbors, choose any four neighbors, we may assume that the vertices have numbers $2, \dots, 5$. Then the Gram matrix of these five vectors has determinant zero if all edges are simple, and negative otherwise.

(5) If there is a vertex of valence 3 such that one of the edges incident to it is a double edge, then the determinant of the 4×4 submatrix is zero. If the multiplicity increases, the determinant becomes negative. If the multiple edge is not incident to a vertex of valence 3, proceed similarly to (3): take the sum of vectors "between" the multiple edge and the vertex of valence 3 and thus reduce the problem to the case above.

(6) Similar to (3): take the sum of vectors "between" two vertices of valence three, and then use (4).

(7) The determinant of the Gram matrix is equal to $pqr\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1\right)$.

(8) The determinants of the corresponding matrices are zero and negative respectively.

□

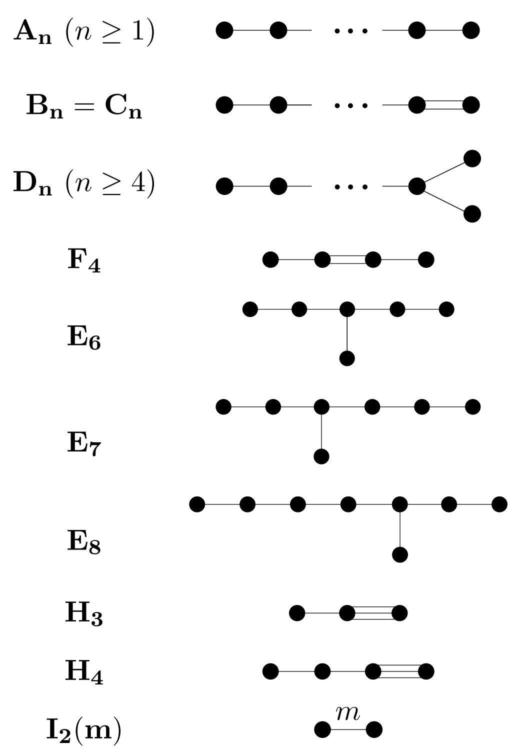
As a result, we get the list of Coxeter diagrams shown in Table 7.1.

Remark. The diagrams without multiple edges (i.e., of types ADE) are called *simply-laced.*

Remark. For each diagram/matrix from Table 7.1 one can construct a system of vectors $\{\alpha_i\}$ with the corresponding Gram matrix. Indeed, as the matrix A is positive definite, there exists a basis in which the quadratic form is given by an identity matrix. Therefore, $A = (A')^t A'$ for some matrix A'. The columns of A' are precisely vectors α_i in the standard orthonormal basis.

Remark. Reflection groups with Coxeter diagram without vertices of valence 3 (i.e., A_n, B_n, F_4, H_3, H_4 and $I_2(m)$) are symmetry groups of *regular polytopes* in $\mathbb{R}^n$ (where a polytope is regular if its symmetry group acts transitively on flags, see HW 11 and 12).

Table 7.1: **Coxeter diagrams.** Connected Coxeter diagrams of finite type



8 Coxeter groups

8.1 Word metric

Coxeter groups can be understood as "abstract reflection groups".

Definition 8.1. Let $S = \{s_1, \dots, s_n\}$. A group with presentation $G = \langle S \mid s_i^2, (s_i s_j)^{m_{ij}} \rangle$, $m_{ij} = 2, \dots, \infty$, is called a *Coxeter group*. A pair (G, S) is called a *Coxeter system*. The number $n = |S|$ is the *rank* of (G, S).

It follows from Theorem 7.7 that every finite reflection group is a Coxeter group. If $m_{ij} = \infty$ then there is no relation between s_i and s_j.

Example. A group $G = \langle s_1, s_2 \mid s_i^2 \rangle$ is an *infinite dihedral group*. It can be understood as the group generated by reflections in two parallel lines in $\mathbb{R}^2$.

Definition 8.2. Let G be any group with a finite generating set S. Assume also that S is symmetric, i.e. for any $s \in S$ the inverse s^{-1} is also contained in S. A *word* is a sequence of elements of S. Any element $g \in G$ can be written as a word $s_{i_1} \dots s_{i_k}$, usually in a non-unique way. For given $g \in G$, the minimal possible k over all words representing g is called *the length* of g and is denoted by $l(g)$. A word $s_{i_1} \dots s_{i_k}$ is *reduced* if for $g = s_{i_1} \dots s_{i_k}$ we have $l(g) = k$. Two words w_1 and w_2 are *equivalent* if they represent the same element of G, notation $w_1 \sim_G w_2$.

Example. Consider $I_2(3) = S_3 = \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^3 \rangle$. A word $s_1 s_2 s_1 s_2$ is not reduced as $s_1 s_2 s_1 s_2 = s_2 s_1$ in S_3, which is shorter. A word $s_2 s_1$ is reduced as the corresponding element cannot be written as a shorter product of generators.

Lemma 8.3. *Let $g, g' \in G$. Then*

(1) $l(gg') \leq l(g) + l(g')$;

(2) $l(g^{-1}) = l(g)$;

(3) $|l(g) - l(g')| \leq l(g'g^{-1})$.

Proof. (1) Let $s_1 \dots s_k$ be a reduced expression for g, and $s_{k+1} \dots s_{k+m}$ be a reduced expression for g'. Then $gg' = s_1 \dots s_{k+m}$, so $l(gg') \leq k + m = l(g) + l(g')$.

(2) Let $s_1 \dots s_k$ be a reduced expression for g. Then $g^{-1} = s_k^{-1} \dots s_1^{-1}$. Since S is symmetric, we have $l(g^{-1}) \leq l(g)$. Therefore, we have $l(g) = l((g^{-1})^{-1}) \leq l(g^{-1})$, and thus $l(g) = l(g^{-1})$.

(3) is left as an exercise (see HW 13.3).

□

Exercise 8.4. Show that $d(g, g') = l(g'g^{-1})$ defines a metric on G.

8.2 Reflections, Exchange Condition and Deleting Condition

Definition 8.5. Let (G, S) be a Coxeter system. A *reflection* is an element of G conjugated to an element of S, the set of reflections is denoted by $R = \{gsg^{-1} \mid s \in S\}$. Elements of S are called *simple reflections.*

If $w = s_1 \dots s_k$ is a word, an *R-sequence* $r(w)$ of w is defined as $r(w) = r_1, \dots, r_k$, where $r_i = (s_1 s_2 \dots s_{i-1}) s_i (s_{i-1} \dots s_1) \in R$.

Remark 8.6. For any $i \leq k$, $s_1 \dots s_i = r_i \dots r_1$.

Example. In $I_2(3)$, $r(s_1 s_2 s_1 s_2) = (s_1, s_1 s_2 s_1, s_1 s_2 s_1 s_2 s_1, s_1 s_2 s_1 s_2 s_1 s_2 s_1) = (s_1, s_1 s_2 s_1, s_2, s_1)$.

Remark. As we have seen while proving Theorem 7.7, in a finite reflection group a word w defines a path going through chambers. Then the reflection sequence $r(w)$ consists of the reflections we need to apply to go from a chamber to the neighboring one.

Exercise 8.7. Show that $r(uv) = (r(u), ur(v)u^{-1})$.

Example. In $I_2(3)$, $r(s_1 s_2) = (s_1, s_1 s_2 s_1) = (r(s_1), s_1 r(s_2) s_1^{-1})$.

Definition 8.8. Given a word w and $r \in R$, define a non-negative integer $n(w, r)$ as the number of appearances of r in $r(w)$.

Theorem 8.9. *Let (G, S) be a Coxeter system, w, w_1, w_2 are words.*

(1) If $w_1 \sim_G w_2$, then $(-1)^{n(w_1, r)} = (-1)^{n(w_2, r)}$ for any $r \in R$.

(2) w is reduced if and only if $n(w, r) \leq 1$ for any $r \in R$.

Proof. By definition, two words w_1 and w_2 are equivalent if they can be taken to each other by a sequence of applications of relations. Every application of a relation v consists of removing or inserting a subword v in a given word, i.e. a word uw is substituted with uvw (or vice versa), where $v = s_i^2$ or $v = (s_i s_j)^{m_{ij}}$. Thus, to prove (1), we need to verify the statement for just one such move (for any relation).

By Exercise 8.7, $r(uw) = (r(u), ur(w)u^{-1})$, so

$$r(uvw) = (r(uv), uvr(w)v^{-1}u^{-1}) = (r(u), ur(v)u^{-1}, uvr(w)v^{-1}u^{-1})$$

Note that $uvr(w)v^{-1}u^{-1}$ coincides with $ur(w)u^{-1}$ as $v \sim_G e$ (and R-sequence is a sequence of group elements, not words). Comparing the sequences above, this implies that we just need to show that the sequence $ur(v)u^{-1}$ contains an even number of any reflection, or, equivalently, this holds for $r(v)$.

If $v = s_i^2$, then $r(v) = (s_i, s_is_is_i) = (s_i, s_i)$.

If $v = (s_is_j)^{m_{ij}}$, then

$$r(v) = (s_i, s_is_js_i, \dots) = (s_i(s_js_i)^k)_{k=0}^{2m_{ij}-1} = ((s_i(s_js_i)^k)_{k=0}^{m_{ij}-1}, (s_i(s_js_i)^k)_{k=m_{ij}}^{2m_{ij}-1}) =$$
$$= ((s_i(s_js_i)^k)_{k=0}^{m_{ij}-1}, (s_i(s_js_i)^k(s_js_i)^{m_{ij}})_{k=0}^{m_{ij}-1})$$

Since $(s_js_i)^{m_{ij}} = e$, we see that every entry in the R-sequence of v appears an even number of times.

To prove (2), assume first that all r_i in the R-sequence of w are distinct. If $s_1 \dots s_k$ is not reduced, then $s_1 \dots s_k \sim_G s_{i_1} \dots s_{i_m}$ for $m < k$. By (1), the R-sequence of $s_{i_1} \dots s_{i_m}$ must contain all k reflections from $r(w)$; however, as $m < k$, this is impossible.

Conversely, let $r_i = r_j$ for $i < j$, where $r(w) = (r_1, \dots, r_k)$. Recall that $r_i = (s_1s_2 \dots s_{i-1})s_i(s_{i-1} \dots s_1)$, and then

$$r_j = (s_1s_2 \dots s_{j-1})s_j(s_{j-1} \dots s_1) = (s_1s_2 \dots s_{i-1})(s_i \dots s_{j-1})s_j(s_{j-1} \dots s_i)(s_{i-1} \dots s_1)$$

Since $r_i = r_j$, we get $s_i \sim_G (s_i \dots s_{j-1})s_j(s_{j-1} \dots s_i)$, which implies $s_i \dots s_j \sim_G s_{i+1} \dots s_{j-1}$, so we can shorten w, and thus w is not reduced.

□

Corollary 8.10. *Let $g \in G$, and let w be any word representing g. Then the number of distinct reflections in $r(w)$ is equal to $l(g)$.*

Remark 8.11. Theorem 8.9 implies that elements of R-sequence of a *reduced* word depend on a group element and not on a word representing it. As it was already pointed out, in case of finite reflection groups the R-sequence has a geometric meaning: given $g \in G$, elements of $r(w)$ (where w is a reduced word representing g) are precisely all reflections whose mirrors separate the chamber gC from the initial chamber C.

Theorem 8.9 (and its proof) has several important consequences.

Corollary 8.12 (Deletion Condition)**.** *If $w = s_1 \dots s_k$ is not reduced, then $w \sim_G = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$ for some $i < j$ (where $\hat{s}$ means that the corresponding letter is removed from the word).*

Corollary 8.13 (Exchange Condition)**.** *Let $w = s_1 \dots s_k$ be reduced, $s \in S$, and $l(sw) < k$. Then $w \sim_G ss_1 \dots \hat{s}_i \dots s_k$ for some i.*

Proof. By the Deletion Condition, $sw \sim_G ss_1 \dots s_k$ with two letters removed. If one of them is s, then $sw \sim_G s_1 \dots \hat{s}_i \dots s_k$, so $w \sim_G ss_1 \dots \hat{s}_i \dots s_k$ as required.

Otherwise, $sw \sim_G ss_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$. Then $w \sim_G s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_k$, which contradicts the fact w is reduced. □

The names "Deletion Condition" and "Exchange Condition" are explained by the following theorem (which we will not prove).

Theorem 8.14. *Let G be a group with a finite generating set S consisting of involutions (i.e. $s^2 = e$ for any $s \in S$. Then TFAE:*

(1) (G, S) is a Coxeter system;

(2) (G, S) satisfies the Deletion Condition;

(3) (G, S) satisfies the Exchange Condition.

8.3 Word problem

Let G be a group with a finite generating set S. Given a word w in the alphabet S, is there an algorithm to determine whether $w \sim_G e$?

In general, the answer to the question above is negative (P. S. Novikov, 1955, proved that there exist groups where the problem is undecidable). However, for Coxeter groups the answer is positive. We will show that any word can be transformed to a reduced word by a bounded number of some elementary operations. Then we show that any two reduced words representing the same element of the group are equivalent under these operations.

Definition 8.15. Let (G, S) a Coxeter system, w is a word. An *elementary M-operation* is either deleting a subword of the form ss or replacing a subword of the form $stst\dots$ of length m_{st} by the subword $tsts\dots$ of the same length (where m_{st} is the order of the element (st)). Clearly, any elementary M-operation preserves the element of the group. A word w is *M-reduced* if it cannot be shortened by a sequence of elementary M-operations. We call such a sequence *M-reduction*. Notice that any reduced word is obviously M-reduced.

We write $w \to w'$ (or $w' \leftarrow w$) if w' can be obtained from w by M-reduction. Notation $w \leftrightarrow w'$ means that $w \to w'$ and $w' \to w$ simultaneously.

Example. In S_3, take a word $s_1s_2s_1s_2s_1$. There are several M-reductions, one of them is shown below:

$$s_1s_2s_1s_2s_1 \to s_2s_1s_2s_2s_1 \to s_2s_1s_1 \to s_2$$

For any element $g \in G$ define $R(g) = \bigcap r(w)$ for all words w representing g.

Lemma 8.16. *Let $s \in S$, and suppose that $s \in R(g)$. Then there exists a reduced word $w = sw'$ representing g.*

Proof. Take any reduced word $w = s_1 \dots s_k$ representing g, and let $r(w) = (r_1, \dots, r_k)$. Since $s \in R(g)$, we have $s = r_i$ for some $i \le k$. Consider the word sw. By Exercise 8.7,

$$r(sw) = \{s, sr(w)s^{-1}\} = \{s, \dots, sr_{i-1}s, s, sr_{i+1}s, \dots\}$$

Therefore, by Theorem 8.9 sw is not reduced. By the Exchange Condition, g is represented by the word $ss_1 \dots \hat{s}_i \dots s_k$, as required. ∎

Lemma 8.17. *Let $w = st\dots$ be a reduced word representing $g \in G$, $s, t \in S$, and suppose that $s, t \in R(g)$. Then there exists reduced words representing g beginning with a subword $sts\dots$ and $tst\dots$ of length m_{st}.*

Proof. Let $w = vu$, where v is a maximal subword of the form $sts\dots$, and denote by q the length of v. Since w is reduced, $q \le m_{st}$. We can assume that $q < m_{st}$, otherwise we have nothing to prove. We can write

$$r(w) = r(vu) = (r(v), vr(u)v^{-1}) = (s, sts, \dots, (st)^{q-1}s, vr(u)v^{-1})$$

Notice that $t \notin r(v)$: if $t = (st)^{p-1}s$ then $(st)^p = 1$ for some $p \le q < m_{st}$, so we would get a contradiction. Since $t \in r(w)$ by the assumption, this implies that $t \in vr(u)v^{-1}$. Since tw is not reduced (see Theorem 8.9), by the Deleting Condition $w \sim_G tvu'$, where u' is obtained from u by omitting one letter. Now, tvu' starts with $tv = tsts\dots$ of length $q+1$. Applying induction, we obtain the statement of the lemma. ∎

Denote by $lt(w)$ the length of a word in (G, S).

Exercise 8.18. (1) If $w \sim_G w'$, then $lt(w) - lt(w') \equiv 0 \bmod 2$.

(2) Let $r \in R$ and $g \in G$. Show that $r \in R(g)$ if and only if $l(rg) < l(g)$.

We can now formulate the main theorem of this section.

Theorem 8.19. *Let w_1 and w_2 be M-reduced words, and $w_1 \sim_G w_2$. Then $lt(w_1) = lt(w_2)$, and $w_1 \leftrightarrow w_2$. In particular, any M-reduced word is reduced.*

Proof. The proof is by induction on $(lt(w_1), lt(w_2))$. We can assume that $lt(w_1) \geq lt(w_2)$.

Case 1: $lt(w_1) > lt(w_2)$.
Let $w_1 = sw_1'$ for some $s \in S$. Then $w_1' \sim_G sw_2$, and w_1' is M-reduced. Denote by w_2' an M-reduced word such that $sw_2 \to w_2'$. By Exercise 8.18, we have $lt(w_1) \geq lt(w_2) + 2$. Thus, we see that $lt(w_1') \geq l(sw_2) \geq l(w_2')$, and $w_1' \sim_G w_2'$. By the induction assumption, $lt(w_1') = lt(w_2')$ and $w_1' \leftrightarrow w_2'$. Therefore, $lt(w_1') = lt(sw_2) = lt(w_2')$, so sw_2 is M-reduced and $sw_2 \leftrightarrow w_1'$. Thus, $w_2 \leftarrow ssw_2 \leftrightarrow sw_1' = w_1$, that contradicts the fact that w_1 is M-reduced.

Case 2: $lt(w_1) = lt(w_2)$, and either w_1 or w_2 is not reduced.
We assume that w_2 is not reduced. Then there exists a reduced word $w_2' \sim_G w_2$, $lt(w_2') < lt(w_2)$. Now we have a pair of M-reduced words $w_2' \sim_G w_1$ with $lt(w_2') < lt(w_1)$, which is impossible by Case 1.

Case 3: w_1 and w_2 are reduced, and $w_1 = sw_1'$, $w_2 = sw_2'$ for some $s \in S$.
In this case we see that $w_1' \leftrightarrow w_2'$ by the induction assumption, so $w_1 \leftrightarrow w_2$.

Case 4: w_1 and w_2 are reduced, and $w_1 = s_1 w_1'$, $w_2 = s_2 w_2'$, $s_1 \neq s_2$.
Let $g \in G$ be the element represented by w_1 and w_2. Then $s_1, s_2 \in R(g)$. By Lemma 8.17, there exists $w_3 \sim_G w_1, w_2$, $w_3 = vu$, where $v = s_1 s_2 \ldots$ has length $m_{s_1 s_2}$. Denote by v' the word $v' = s_2 s_1 \ldots$ of length $m_{s_1 s_2}$, let $w_4 = v'u$. By Case 3, $w_1 \leftrightarrow w_3 \leftrightarrow w_4 \leftrightarrow w_2$, which completes the proof.

□

Corollary 8.20. *w represents $e \in G$ if and only if $w \to \emptyset$.*

As the number of M-reductions is finite, this provides an algorithm.

8.4 Coxeter groups and reflection groups

We have already seen that finite reflection groups are Coxeter groups.

Conversely, let (G, S) be a Coxeter system. We construct a discrete linear group in real vector space of dimension $|S|$, such that for any two elements $s, t \in S$ the order of (st) is equal to m_{st}, and $s^2 = e$.

Let V be a real vector space with basis $\{e_s,\ s \in S\}$. Define a symmetric bilinear form on V by

$$(e_s, e_t) = -\cos(\pi/m_{st})$$

In particular, $(e_s, e_s) = 1$, and $(e_s, e_t) = -1$ if $m_{st} = \infty$.

Exercise 8.21. Show that $\forall s \in S\ \ V = \langle e_s \rangle \oplus \langle e_s \rangle^{\perp}$.

Denote by H_s the hyperplane $\langle e_s \rangle^{\perp}$. A *reflection* in H_s is given by

$$r_s(x) = x - 2(x, e_s)e_s$$

Now let $s, t \in S$, $s \neq t$.

If $m_{st} \neq \infty$, the bilinear form is positive definite on the plane $\langle e_s, e_t \rangle$. Thus, $V = \langle e_s, e_t \rangle \oplus \langle e_s, e_t \rangle^{\perp}$. Reflections r_s and r_t fix $\langle e_s, e_t \rangle^{\perp}$, and generate a dihedral group on $\langle e_s, e_t \rangle$. Therefore, the order of (st) equals m_{st}.

If $m_{st} = \infty$, $r_s(e_t) = e_t + 2e_s$ and $r_t(e_s) = e_s + 2e_t$. Thus, the restriction of r_s, r_t and $r_s r_t$ onto the plane $\langle e_s, e_t \rangle$ is given by matrices

$$r_s \sim \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \qquad r_t \sim \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \qquad r_s r_t \sim \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

But the latter matrix has determinant 1 and trace 2 (and is not the identity matrix), so it is conjugated to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which has infinite order. Therefore, the order of (st) is infinite.

Definition 8.22. Let (G, S) be a Coxeter system. A *Coxeter diagram* of G is a (multi)graph defined as follows:

- vertices are indexed by s_i (or simply by natural numbers $1, \ldots, n$);
- vertices i and j are joined by $m_{ij} - 2$ edges (or a simple edge with weight m_{ij}), where $(s_i s_j)^{m_{ij}} = e$, if $m_{ij} \in \mathbb{N}$, and by a thick edge if $m_{ij} = \infty$.

Remark. The definition agrees with the definition of the Coxeter diagram of a finite reflection group.

Example. A Coxeter diagram of an infinite dihedral group consists of one thick edge.

Theorem 8.23. *Let (G, S) be a Coxeter system. Then G is finite if and only if the bilinear form on $\mathbb{R}^{|S|}$ defined above is positive definite.*

Proof. If the form is positive definite, then the Coxeter diagram coincides with one of the Coxeter diagrams of finite reflection groups. In particular, G has the same presentation as one of the finite reflection groups, so it is finite.

Conversely, if the form is not positive definite, then the Coxeter diagram contains one of "prohibited" subdiagrams from Lemma 7.19 (or a thick edge which corresponds to a pair of simple reflections such that its product has an infinite order). According to HW 15.2, the corresponding subgroup is also a Coxeter group with the presentation defined by the subdiagram. One now needs to find an element of infinite order in each of the corresponding Coxeter groups, which we leave as an exercise.

□

We will see a more conceptual proof of the theorem later on.

Example. Consider a subdiagram consisting of two multiple edges, i.e. a group with presentation $\langle s_1, s_2, s_3 \mid s_i^2, (s_1 s_2)^k, (s_2 s_3)^m, (s_1 s_3)^2 \rangle$ with $k, m \geq 4$. Let $w = s_1 s_2 s_3$. Then every word w^k is M-reduced: the only M-operation applicable to it is interchanging $s_1 s_3$ with $s_3 s_1$, which does not change the length. In particular, the corresponding group element is of infinite order.

9 Root systems

9.1 Definitions

Definition 9.1. A set $\Delta \subset \mathbb{R}^n$ is a *root system* of rank n if the following conditions hold:

(1) Δ is finite and spans $\mathbb{R}^n$;

(2) the expression $\langle \alpha \mid \beta \rangle \overset{\text{def}}{=} \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$;

(3) $r_\alpha(\Delta) = \Delta$ for any $\alpha \in \Delta$.

A root system is *reduced* if the following holds:

(4) if $\alpha \in \Delta$ and $c\alpha \in \Delta$ for some $c \in \mathbb{R}$ then $c = \pm 1$.

By default, we will assume that all root systems are reduced.

Example 9.2. Take a reflection group of type A_2 and normalize *inward* normals α, β to the walls of a chamber to have length $\sqrt{2}$. Then $(\alpha, \beta) = -1$, so $\langle \alpha \mid \beta \rangle = -1$. Now, a normal to the third mirror is $\alpha + \beta$, $\|\alpha + \beta\|^2 = (\alpha + \beta, \alpha + \beta) = 2$ as well. In particular, $\langle \alpha + \beta \mid \alpha \rangle = \langle \alpha \mid \alpha \rangle + \langle \beta \mid \alpha \rangle = 1$. So, we see that the collection of six vectors $\pm\alpha, \pm\beta, \pm(\alpha + \beta)$ is a root system.

Lemma 9.3. *Let $\angle(\alpha, \beta) = \varphi$. Then $\langle \alpha \mid \beta \rangle \langle \beta \mid \alpha \rangle = 4\cos^2\varphi$. In particular, $\varphi \in \{\frac{k\pi}{2}, \frac{k\pi}{3}, \frac{k\pi}{4}, \frac{k\pi}{6}\}$.*

This can be proved by an elementary computation.

Definition 9.4. Given $\alpha \in \Delta$, denote $H_\alpha = \alpha^\perp$ the mirror of the reflection r_α. A connected component of $\mathbb{R}^n \setminus \bigcup\limits_{\alpha \in \Delta} H_\alpha$ is called a *Weyl chamber* of Δ.

Given a Weyl chamber C, denote by $\Pi = \Pi(C)$ the set of inward normals to the walls of Δ lying in Δ. Π is called a *set of simple roots* of Δ (defined by C). Then $C = \{v \in \mathbb{R}^n \mid (\alpha, v) < 0 \, \forall \, \alpha \in \Pi\}$.

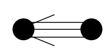
The group $W = W(\Delta)$ generated by all reflections r_α, $\alpha \in \Delta$, is called the *Weyl group* of Δ. If we denote $S = \{r_\alpha \mid \alpha \in \Pi\}$, then the pair (W, S) is a Coxeter system. All sets of simple roots are equivalent under action of W.

Given Π, the matrix $A = (a_{ij}) = (\langle \alpha_i \mid \alpha_j \rangle)$ is called a *Cartan matrix* of Δ.

Example 9.5. Weyl group of type G_2 is generated by two reflections in lines forming an angle $\pi/6$. Since the minimal angle between mirrors is $\varphi = \pi/6$, we have $4\cos^2\varphi = 3$, and thus $\langle \alpha \mid \beta \rangle \langle \beta \mid \alpha \rangle = 3$, where α, β simple roots. This implies that $\langle \alpha \mid \beta \rangle = -1$ and $\langle \beta \mid \alpha \rangle = -3$ (or vice versa). This, in its turn, implies that $\frac{(\beta,\beta)}{(\alpha,\alpha)} = 3$.

Therefore, we may take the root system consisting of six vectors $e^{\frac{k\pi i}{3}}$ of unit length and of six vectors $\sqrt{3}e^{\frac{\pi i}{6} + \frac{k\pi i}{3}}$ of length $\sqrt{3}$. The Cartan matrix is $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

Definition 9.6. *Dynkin diagram* of a root system Δ is a graph with oriented (multiple) edges. Vertices are indexed by elements of Π; if $|a_{ij}| \geq |a_{ji}|$, then there are $|a_{ij}|$ edges between i and j; the edges are oriented from i to j if $a_{ij} \neq a_{ji}$, otherwise they are non-oriented.

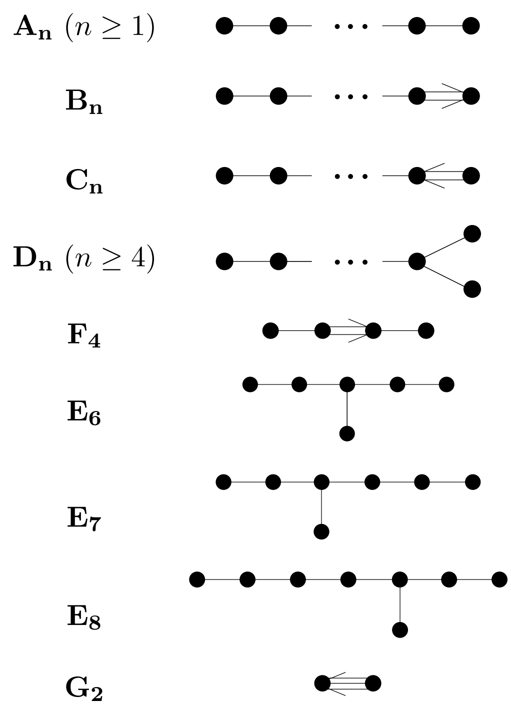
Example 9.7. The Dynkin diagram for the root system of type G_2 is 

9.2 Classification

Lemma 9.8. *(a) A set of simple roots Π defines Δ uniquely.*

(b) Let G be a finite reflection group, let $\alpha_1, \ldots, \alpha_n$ be inward normals to the walls of a chamber C of G, and assume that $\langle \alpha_i \mid \alpha_j \rangle \in \mathbb{Z}$ for all i, j. Then there exists a unique root system Δ with $W(\Delta) = G$ and $\Pi = \{\alpha_i\}$.

Table 9.1: **Dynkin diagrams.** Dynkin diagrams of finite irreducible root systems



Proof. To prove (a) we just need to observe that $\Delta = W\Pi$.

To prove (b), consider a *lattice* $L = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n = \{c_1\alpha_1 + \cdots + c_n\alpha_n \mid a_i \in \mathbb{Z}\}$. Since $r_{\alpha_j}(\alpha_i) = \alpha_i - \langle \alpha_i \mid \alpha_j \rangle \alpha_j$ and $\langle \alpha_i \mid \alpha_j \rangle \in \mathbb{Z}$, L is invariant with respect to all r_{α_j}, and thus is invariant with respect to G.

Notice also that the set $G\{\alpha_i\}$ contains normals to all mirrors of G. Define Δ to be the set of all *primitive* vectors of L orthogonal to mirrors of G (a vector v is primitive in L if $v \neq cv'$ for $v' \in L$ and $|c| > 1$), i.e.

$$\Delta = \{\alpha \in L \mid r_\alpha \in G,\ \alpha \neq cv\ \forall v \in L, c \in \mathbb{Z}_{>1}\}$$

Then Δ is finite, invariant with respect to $W = G$, reduced, and for every $\alpha, \beta \in \Delta$ we have $r_\alpha(\beta) = \beta - \langle \beta \mid \alpha \rangle \alpha \in L$, which implies $\langle \beta \mid \alpha \rangle \alpha \in L$, and thus $\langle \beta \mid \alpha \rangle \in \mathbb{Z}$.

□

Corollary 9.9 (Classification of irreducible root systems)**.** *A root system is* irreducible *if its Dynkin diagram is connected (or, equivalently, if its Weyl group is irreducible). Table 9.1 contains Dynkin diagrams of all irreducible root systems.*

9.3 Structure of finite root systems

Definition 9.10. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots, and let $\alpha = \sum c_i\alpha_i \in \Delta$. A root α is *positive* if all $c_i \geq 0$, and *negative* if all $c_i \leq 0$. The set of positive roots is denoted by Δ^+, and the set of negative roots is denoted by Δ^-.

Example 9.11. All simple roots are positive.

Theorem 9.12. $\Delta = \Delta^+ \cup \Delta^-$.

Proof. Let C be the Weyl chamber such that $\Pi = \Pi(C)$. Take any $\alpha \in \Delta$, then either $(\alpha, v) > 0$ for any $v \in C$, or $(\alpha, v) < 0$ for any $v \in C$.

The theorem follows from the following lemma.

Lemma 9.13. *Let $v \in C$, $\alpha \in \Delta$. If $(\alpha, v) > 0$ then $\alpha \in \Delta^+$, and if $(\alpha, v) < 0$ then $\alpha \in \Delta^-$.*

Proof of Lemma 9.13. Define the dual basis $\{\omega_i\}$ to $\{\alpha_i\}$ by $(\alpha_i, \omega_j) = 1$ if $i = j$ and $(\alpha_i, \omega_j) = 0$ otherwise. Then ω_i is orthogonal to all but one simple roots and $(\omega_i, \alpha_i) = 1 > 0$, so ω_i is a one-dimensional face of C. Thus, C can be written as $C = \{v \in \mathbb{R}^n \mid v = a_1\omega_1 + \cdots + a_n\omega_n,\ a_i > 0\}$. In particular, given $u \in \mathbb{R}^n$, we have $(u, v) < 0$ for any $v \in C$ if and only if $(u, \omega_i) \leq 0$ for all i (and similar for positive scalar product).

Now, let $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n \in \Delta$, $v \in C$. Since (α, u) has the same sign for all $u \in C$, we have $(\alpha, v) < 0$ if and only $(\alpha, \omega_i) < 0$ for all i. But $(\alpha, \omega_i) = c_i$, so $(\alpha, v) < 0$ if and only if $c_i < 0$ for all i, i.e. $\alpha \in \Delta^-$. The other case is identical.

□

The lemma shows that $\Delta^+ = \{\alpha \in \Delta \mid (\alpha, v) > 0\ \forall v \in C\}$ and $\Delta^- = \{\alpha \in \Delta \mid (\alpha, v) < 0\ \forall v \in C\}$, which proves the theorem.

□

Remark 9.14. The definition of positivity clearly depends on the chamber. Theorem 9.12 shows that positivity can also be defined by (almost) any element f_v of the dual vector space, where $f_v(\alpha) = (v, \alpha)$: α is positive if and only if $f_v(a) > 0$ – this is equivalent to α being positive with respect to the chamber containing v.

Exercise 9.15. Let $\alpha, \beta \in \Delta$.

(a) If $(\alpha, \beta) > 0$ then either $\alpha - \beta \in \Delta$ or $\alpha = \beta$.

(b) If $(\alpha, \beta) < 0$ then either $\alpha + \beta \in \Delta$ or $\alpha = -\beta$.

9.4 Root poset and the highest root

Definition 9.16. Let $\alpha, \beta \in \Delta^+$. We say that $\alpha \geq \beta$ if $\alpha - \beta = \sum c_i\alpha_i$ with $\alpha_i \in \Pi$, all $c_i \geq 0$. This supplies Δ^+ with a partial order, and thus defines the *root poset*.

Exercise. Draw the Hasse diagram of the root poset of type A_2.

Lemma 9.17. *Let C be a Weyl chamber of Δ, $\Pi = \{\alpha_i\}$ be simple roots, and $\overline{C}$ be the closure of C. There exists a unique maximal element $\widetilde{\alpha}_0$ of the root poset, i.e. $\widetilde{\alpha}_0 \geq \alpha$ for every $\alpha \in \Delta^+$ (it is called the* highest root*). Moreover, $\widetilde{\alpha}_0 \in \overline{C}$, and $\widetilde{\alpha}_0 = \sum\limits_{\alpha_i \in \Pi} c_i\alpha_i$ with all $c_i > 0$.*

Proof. First, let α be a maximal element of the root poset. If $(\alpha, \alpha_i) < 0$ for some i, then by Exercise 9.15 $\alpha + \alpha_i \in \Delta$ (or $\alpha = -\alpha_i$ which cannot happen as $\alpha \in \Delta^+$), which contradicts maximality. Therefore, $(\alpha, \alpha_i) \geq 0$ for all i, and thus $\alpha \in \overline{C}$.

Further, write $\alpha = \sum\limits_{\alpha_i \in \Pi} c_i\alpha_i$, $c_i \geq 0$. Denote $I = \{i \mid c_i > 0\}$ and $J = \{j \mid c_j = 0\}$, and assume $J \neq \emptyset$. As the Dynkin diagram is connected, there exist neighboring nodes $i \in I$ and $j \in J$, so $(\alpha_i, \alpha_j) \neq 0$. Then $(\alpha, \alpha_j) < 0$, which is not the case as we have proved above. Therefore, J is empty, and thus all coefficients of α are positive.

Finally, assume there are two maximal elements in the root poset, α and β. Then $(\alpha, \beta) = \sum c_i(\alpha_i, \beta)$. As we have proved, for every i we have $(\alpha_i, \beta) > 0$ and $c_i > 0$, so $(\alpha, \beta) > 0$. By Exercise 9.15, either

$\alpha = \beta$ (and we are done) or $\alpha - \beta \in \Delta$. In the latter case, either $\alpha - \beta \in \Delta^+$ or $\beta - \alpha \in \Delta^+$, so either $\alpha > \beta$ or $\beta > \alpha$, and thus we come to a contradiction.

□

Remark 9.18. It follows from the classification of root systems (see Section 9.2) that a root system can contain roots of at most two distinct lengths. If all roots are of the same length, the root system is called *simply-laced*. Otherwise, a roots system contains roots of two lengths, *long* roots and *short* roots. For root systems of types B_n, C_n and F_4, the ratio of lengths of long and short roots is $\sqrt{2}$, while for G_2 it is $\sqrt{3}$.

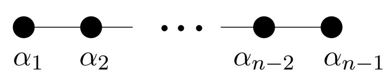
In particular, the highest root is always long (see HW 18.3).

10 Construction of finite root systems

Let $e_1, \dots, e_n$ be an orthonormal basis of $\mathbb{R}^n$.

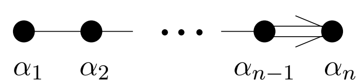
10.1 Classical series: ABCD

10.1.1 A_{n-1}



Consider the hyperplane $H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i = 0\} \simeq \mathbb{R}^{n-1}$. Define $\Delta = \{e_i - e_j\}$, $i \neq j$. Then $r_{e_i - e_j}$ preserves H and acts by permutation of i-th and j-th coordinates. $W \simeq S_n$ (and thus $|W| = n!$), simple roots are $\Pi = \{\alpha_i = e_i - e_{i+1}\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in H \mid x_1 > x_2 > \cdots > x_n\}$, so positive roots are $\{e_i - e_j\}$ for $i < j$. The highest root is $\sum\limits_{\alpha_i \in \Pi} \alpha_i = e_1 - e_n$.

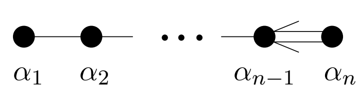
10.1.2 B_n



Define $\Delta = \{\pm e_i \pm e_j, \pm e_i\}$, $i < j$. Then $r_{e_i - e_j}$ acts by permutation of i-th and j-th coordinates, and r_{e_i} acts by change of sign of i-th coordinate. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n > 0\}$, so positive roots are $\{e_i \pm e_j\}$ for $i < j$ and $\{e_i\}$. The highest root is $\alpha_1 + \sum\limits_{i=2}^{n} 2\alpha_i = e_1 + e_2$.

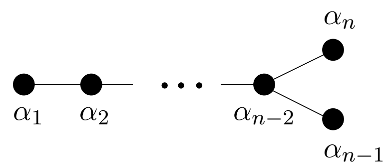
To determine the order of the Weyl group W, observe that if an element of W fixes the basis $\{e_i\}$ then it is identity, and thus we just need to count the bases W sends $\{e_i\}$ to. Every reflection in W is either a permutation of the basis vectors, or a change of sign of a basis vector, or a combination of these ($r_{e_i + e_j}$ takes e_i to $-e_j$ and e_j to $-e_i$ leaving all others intact). Then any element of W is also a permutation of the basis vectors and a change of signs of some basis vectors. Applying reflections in long roots, we can get any permutation, and applying reflections in short roots we can get any sign change. Therefore, there are $n!2^n$ configurations we can get, so $|W| = n!2^n$.

10.1.3 C_n



Root system C_n differs from B_n by the lengths of roots only. We can take $\Delta = \{\pm e_i \pm e_j, \pm 2e_i\}$, $i < j$. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n\}$, positive roots are $\{e_i \pm e_j\}$ for $i < j$ and $\{2e_i\}$. The highest root is $\sum_{i=1}^{n-1} 2\alpha_i + \alpha_n = 2e_1$. The Weyl group (and Weyl chamber) coincides with the one for B_n.

10.1.4 D_n



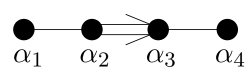
Define $\Delta = \{\pm e_i \pm e_j\}$, $i < j$ – this is precisely the set of long roots of B_n. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in \mathbb{R}^n \mid x_1 > x_2 > \cdots > x_n, x_{n_1} + x_n > 0\}$, positive roots are $\{e_i \pm e_j\}$ for $i < j$. The highest root is $\alpha_1 + \sum_{i=2}^{n-2} 2\alpha_i + \alpha_{n-1} + \alpha_n = e_1 + e_2$.

The order of the Weyl group W can be determined similarly to the B_n case: every reflection is either a transposition of two basis vectors, or a transposition composed with the change of both signs. Thus, we can get any permutation, but the number of changes of sign should be even. This implies that there are $n!2^{n-1}$ configurations we can get, so $|W| = n!2^{n-1}$.

Another way to find the order of the group is to look at the Weyl chamber: it is a union of two copies of the Weyl chamber of B_n (reflected in the hyperplane $x_n = 0$). Therefore, $[W(B_n) : W(D_n)] = 2$, and the result follows.

10.2 Exceptional root systems: EFG

10.2.1 F_4



Consider the lattice $L \subset \mathbb{R}^4$ defined as follows: $L = \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, e_3, e_4, \frac{1}{2}(e_1 + e_2 + e_3 + e_4)\}$. In other words, L consists of all linear combinations of five vectors above with integer coefficients. Define

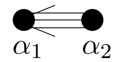
$$\Delta = \{v \in L \mid (v, v) = 1 \text{ or } (v, v) = 2\} = \{\pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4), i < j\}$$

The reflections in vectors of Δ preserve L, and thus the whole group W preserves L, which implies W preserves Δ. Simple roots are $\Pi = \{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in \mathbb{R}^4 \mid x_2 > x_3 > x_4 > 0, x_1 > x_2 + x_3 + x_4\}$, so positive roots are $\{e_i \pm e_j\}$ for $i < j$, $\{e_i\}$, $\{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$. The highest root is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = e_1 + e_2$.

To find the order of the Weyl group observe that W acts transitively on long roots (cf. HW 13.1). There are 24 long roots, so we are left to find the order of the stabilizer of one of them. Consider the highest root $\widetilde{\alpha}_0 = e_1 + e_2$, it is orthogonal to all simple roots except for α_1. Due to HW 16.4, its stabilizer is generated by all reflections with respect to simple roots lying in $\widetilde{\alpha}_0^{\perp}$, which in our case generate Weyl group C_3. Thus, $|W| = 24|W(B_3)| = 24 \cdot 48 = 1152$.

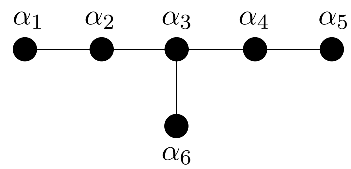
Remark. $W(F_4)$ is a symmetry group of a regular 4-dimensional polytope, a 24-*cell*.

10.2.2 G_2



Here the Weyl group is a dihedral group of type $I_2(6)$, we can associate $\mathbb{R}^2$ with $\mathbb{C}$. Then $\Delta = \{\exp(\frac{k\pi i}{3}), \sqrt{3}\exp(\frac{\pi i}{6} + \frac{k\pi i}{3}), k = 0, \dots 5\}$. The simple roots are $\alpha_1 = 1, \alpha_2 = \sqrt{3}\exp(\frac{5\pi i}{6})$, the positive roots are all with argument in $[0, \pi)$. The highest root is $3\alpha_1 + 2\alpha_2 = \sqrt{3}i$.

10.2.3 E_6



Consider $\mathbb{R}^7$ with orthonormal basis $\{e_1, \dots, e_6, e\}$. Let $H = \{(x_1, \dots, x_6, y) \in \mathbb{R}^7 \mid \sum x_i = 0\} \simeq \mathbb{R}^6$. Define

$$L = \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, e_3, e_4, e_5, e_6, \sqrt{2}e, \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + \sqrt{2}e)\} \cap H,$$

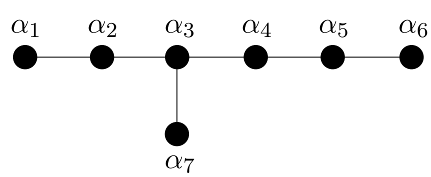
and let

$$\Delta = \{v \in L \mid (v, v) = 2\} = \{e_i - e_j, \pm\sqrt{2}e, \frac{1}{2}(e_i + e_j + e_k - e_p - e_q - e_r \pm \sqrt{2}e)\},$$

where all indices are distinct. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \dots, \alpha_5 = e_5 - e_6, \alpha_6 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + \sqrt{2}e)\}$. The corresponding Weyl chamber is given by inequalities $C = \{(x, y) \in H \mid x_1 > x_2 > x_3 > x_4 > x_5 > x_6, \sqrt{2}y + x_4 + x_5 + x_6 > x_1 + x_2 + x_3\}$, so positive roots are $\{e_i - e_j\}$ for $i < j$, $\sqrt{2}e$, $\{\frac{1}{2}(e_i + e_j + e_k - e_p - e_q - e_r + \sqrt{2}e)\}$. The highest root is $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 = \sqrt{2}e$.

There are 72 roots, the highest root is orthogonal to simple roots forming a subdiagram of type A_5. Therefore, $|W| = 72 \cdot 6!$.

10.2.4 E_7



Let $H = \{x \in \mathbb{R}^8 \mid \sum x_i = 0\} \simeq \mathbb{R}^7$. Define

$$L = \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8)\} \cap H,$$

and let

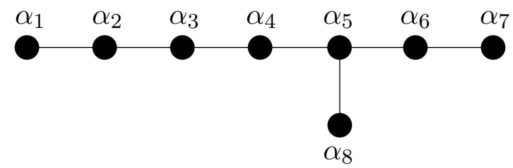
$$\Delta = \{v \in L \mid (v, v) = 2\} = \{e_i - e_j, \frac{1}{2}(e_i + e_j + e_k + e_l - e_p - e_q - e_r - e_s)\},$$

where all indices are distinct. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \dots, \alpha_6 = e_6 - e_7,$
$\alpha_7 = \frac{1}{2}(-e_1 - e_2 - e_3 + e_4 + e_5 + e_6 + e_7 - e_8)\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in H \mid x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_7, x_4 + x_5 + x_6 + x_7 > x_1 + x_2 + x_3 + x_8\}$, positive roots are $\{e_i - e_j\}$ for $i < j$, $\{\frac{1}{2}(e_i + e_j + e_k + e_l - e_p - e_q - e_r - e_8)\}$.

The highest root is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 = e_1 - e_8$.

There are 126 roots, the highest root is orthogonal to simple roots forming a subdiagram of type D_6. Therefore, $|W| = 126 \cdot 6! 2^5$.

10.2.5 E_8



Let $H = \{x \in \mathbb{R}^9 \mid \sum x_i = 0\} \simeq \mathbb{R}^8$. Define

$$L = \mathrm{span}_{\mathbb{Z}}\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, \frac{1}{3}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 + e_9)\} \cap H,$$

and let

$$\Delta = \{v \in L \mid (v, v) = 2\} = \{e_i - e_j, \pm\frac{1}{3}(2(e_i + e_j + e_k) - (e_l + e_p + e_q + e_r + e_s + e_t))\},$$

where all indices are distinct. Simple roots are $\Pi = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_7 = e_7 - e_8,$ $\alpha_8 = \frac{1}{3}(-e_1 - e_2 - e_3 - e_4 - e_5 + 2(e_6 + e_7 + e_8) - e_9)\}$. The corresponding Weyl chamber is given by inequalities $C = \{x \in H \mid x_1 > x_2 > x_3 > x_4 > x_5 > x_6 > x_7 > x_8, 2(x_6 + x_7 + x_8) > x_1 + x_2 + x_3 + x_4 + x_5 + x_9\}$, positive roots are $\{e_i - e_j\}$ for $i < j$, $\{\frac{1}{2}(2(e_i + e_j + e_k) - (e_l + e_p + e_q + e_r + e_s + e_9))\}$.

The highest root is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 + 3\alpha_8 = e_1 - e_9$.

There are 240 roots, the highest root is orthogonal to simple roots forming a subdiagram of type E_7. Therefore, $|W| = 240 \cdot 126 \cdot 6! 2^5$.

10.3 Bonus track: Finite Coxeter groups that are not Weyl groups

Remark. By dropping the condition $\langle \alpha \mid \beta \rangle \in \mathbb{Z}$ for $\alpha, \beta \in \Delta$ in the definition of a root system we get a definition of a *non-crystallographic* root system. In particular, the set of unit normal vectors to mirrors of all reflections of a finite reflection group satisfies the definition. Most of the results previously proved for root systems hold in these more general settings as well. In particular, everything in Section 9.3 (and HW 16.1) holds.

The construction of dihedral groups is clear. Here we show how to construct groups H_3 and H_4.

10.3.1 H_4

There are several explicit constructions of H_4. The one below follows Lusztig, Scherbak, and Moody–Patera.

Consider the Weyl group of type E_8 with simple reflections $s_1, \ldots, s_4, t_1, \ldots, t_4$, see Fig. 10.1. Denote $r_i = s_i t_i$, and consider group Γ generated by r_1, r_2, r_3, r_4. Clearly, elements r_i are involutions, and r_i commutes with r_j for $|i - j| > 1$. Also, since s_i commutes with t_i, $(r_1 r_2)^3 = (s_1 t_1 s_2 t_2)^3 = (s_1 s_2)^3 (t_1 t_2)^3 = e$, and similarly $(r_2 r_3)^3 = e$. Finally, $r_3 r_4 = s_3 t_3 t_4 s_4$ can be considered as an element of the group A_4, it is easy to check it has order 5 (see also Section 11).

Therefore, generators of Γ satisfy all relations of H_4, and thus there is a surjective homomorphism $H_4 \to \Gamma$.

To see that the kernel is trivial, we interpret r_i as a reflection in the ring of *quaternions* which is isomorphic to $\mathbb{R}^4$ as a vector space over $\mathbb{R}$.

Let $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ be an orthonormal basis of $\mathbb{R}^4$. Define an operation of (non-commutative!) multiplication on the basis as follows:

$$1 \cdot x = x, \quad , \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \quad \boldsymbol{i} \cdot \boldsymbol{j} = -\boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{k}, \quad \boldsymbol{j} \cdot \boldsymbol{k} = -\boldsymbol{k} \cdot \boldsymbol{j} = \boldsymbol{i}, \quad \boldsymbol{k} \cdot \boldsymbol{i} = -\boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j},$$

Figure 10.1: Construction of the group of type H_4 as a subgroup of the Weyl group E_8

and then extend this to the whole $\mathbb{R}^4$ by linearity. We get the algebra of quaternions $\mathbb{H}$. An element of $\mathbb{H}$ has the form $x = (x_1, x_2, x_3, x_4) = x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k}$, $x_l \in \mathbb{R}$. There is a natural operation of conjugation: $\overline{x_1 + x_2\boldsymbol{i} + x_3\boldsymbol{j} + x_4\boldsymbol{k}} = x_1 - x_2\boldsymbol{i} - x_3\boldsymbol{j} - x_4\boldsymbol{k}$. Then the Euclidean norm $\|x\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ can be written as $\|x\|^2 = x\bar{x}$. In particular, the unit sphere in $\mathbb{R}^4$ is $U = \{x \in \mathbb{H} \mid \|x\|^2 = 1\}$. Elements of U are called *unit quaternions.*

The standard dot product on $\mathbb{R}^4$ can also be written in these terms: $(x, y) = \frac{1}{2}(x\bar{y} + y\bar{x})$ (check this!).

Exercise. Let $x \in U$ be a unit quaternion, and $v \in \mathbb{H}$. Show that the reflection of v in x (as vectors in $\mathbb{R}^4$) is given by formula $r_x(v) = -x\bar{v}x$.

Denote $\varphi = 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}$, note that $\varphi^2 = \varphi + 1$. Consider all unit quaternions of types

$(\pm 1, 0, 0, 0)$ with all permutations;

$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$;

$\frac{1}{2}(0, \pm 1, \pm\varphi, \pm(1 - \varphi))$ with all *even* permutations.

Exercise. There are 120 quaternions of types above, and they form a group with respect to multiplication. We denote this set by I.

Now, let us construct a linear map from $\mathbb{R}^8$ to $\mathbb{H}$ as follows. Let α_i be simple roots of the root system E_8 corresponding to simple reflections s_i, and let α_i' be the simple roots corresponding to simple reflections t_i. Define unit quaternions $\beta_1, \dots, \beta_4$ as follows:

$$\beta_1 = \frac{1}{2}(\varphi - 1, -\varphi, 0, -1), \quad \beta_2 = \frac{1}{2}(0, \varphi - 1, -\varphi, 1), \quad \beta_3 = \frac{1}{2}(0, 1, \varphi - 1, -\varphi), \quad \beta_4 = \frac{1}{2}(0, -1, \varphi - 1, \varphi),$$

and then define a linear map $\mathbb{R}^8 \to \mathbb{H}$ by

$$f(\alpha_i) = \beta_i, \qquad f(\alpha_i') = \varphi\beta_i$$

Exercise. The map f takes $s_i t_i$ to precisely r_{β_i}. In other words, for any $p \in \mathbb{R}^8$ we have

$$f(s_i t_i(p)) = r_{\beta_i}(f(p))$$

Exercise. The Gram matrix of vectors β_i is precisely the matrix defined by the Coxeter diagram H_4.

The two exercises above complete the construction. In particular, I is precisely the set of vectors orthogonal to mirrors of H_4, and f takes all 240 roots of E_8 to $I \cup \varphi I$.

More details can be found in the paper by R. V. Moody and J. Patera "Quasicrystals and icosians", J. Phys. A 26 (1993), 2829–2853.

To compute the order of the group observe that all simple reflections are conjugated to each other, and thus W acts transitively on all 120 elements of I. The vector $(1, 0, 0, 0)$ is orthogonal to $\beta_2, \beta_3, \beta_4$, and thus it belongs to $\overline{C}_0$ and its stabilizer is generated by r_2, r_3, r_4, which generate a group of type H_3 of order 120 (see next section). Therefore, $|W| = 120^2 = 14400$.

Remark. $W(H_4)$ is a symmetry group of two regular 4-dimensional polytopes, a 120-*cell* and a 600-*cell.*

10.3.2 H_3

The construction follows from the previous section: we consider subspace H of $\mathbb{R}^8$ spanned by roots α_i, α'_i for $i = 2, 3, 4$. Then we get $H \simeq \mathbb{R}^6$ with a root system of type D_6; H is taken by f to the subspace of $\mathbb{H}$ with zero first coordinate (such quaternions are called *pure*), the image of the root system D_6 is the set of vectors orthogonal to mirrors of H_3 (and their φ-multiples).

The order of the group can be computed as follows. There are 30 elements of I with zero first coordinate, which implies that there are 30 unit vectors orthogonal to mirrors (and 15 reflections), the group acts transitively on them. Vector $(0, \varphi, 1, \varphi - 1)$ is orthogonal to β_2 and β_4, so its stabilizer is generated by two commuting reflections r_2 and r_4. Therefore, $|W| = 30 \cdot 4 = 120$.

$W(H_3)$ is a symmetry group of two regular 3-dimensional polytopes, an icosahedron and a dodecahedron.

10.4 Affine Dynkin diagrams

Let Δ be an irreducible root system, $\Pi = \{\alpha_i\}$ is a set of simple roots, $\widetilde{\alpha}_0$ is the highest root. The root $\alpha_0 = -\widetilde{\alpha}_0$ is called the *lowest* root.

Definition 10.1. The set $\widehat{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ is an *extended* set of simple roots. The corresponding Dynkin diagram with $n + 1$ nodes is an *extended Dynkin diagram*, or an *affine Dynkin diagram*.

The list of affine Dynkin diagrams is shown in Table 10.1.

Remark 10.2. We are now able to interpret the finiteness of the reflection game (see Section 6). The graph corresponds to a quadratic form, and the initial configuration is one basis vector α_i. A configuration is a linear combination $\sum c_i\alpha_i$. A vertex k is unstable if $(\alpha_k, \sum c_i\alpha_i) < 0$. A move in an unstable vertex k is a reflection of the configuration $\sum c_i\alpha_i$ in vector α_k. In case of Dynkin diagrams, the game ends when we get the highest root. If a (simply-laced) graph is not a Dynkin diagram, then it contains an affine Dynkin diagram as a subgraph; the game is infinite on any affine Dynkin diagram.

Definition 10.3. A *generalized Cartan matrix* $A = (a_{ij})$ is an integer $n \times n$ matrix satisfying the following properties: $a_{ii} = 2$, $a_{ij} \le 0$ for $i \ne j$, and $a_{ij} = 0$ implies $a_{ji} = 0$.

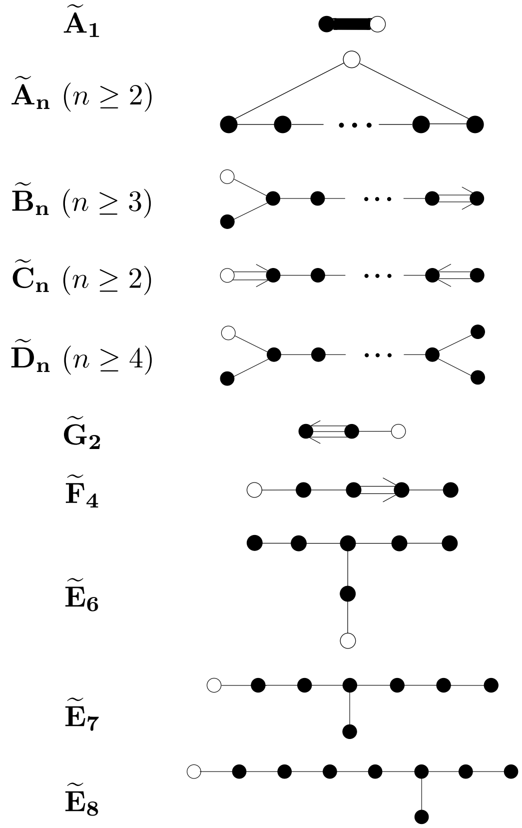
A is of *finite type* (respectively, *affine type* and *indefinite type*) if there exists a vector $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ with all $u_i > 0$ such that Au also has all positive coordinates (respectively, $Au = 0$ and Au has all negative coordinates).

Remark 10.4. Indecomposable generalized Cartan matrices (GCM for short) can be of finite, affine or indefinite type only. If GCM is symmetric, then it is of finite type if and only if it is positive definite, and it is of affine type if and only if it is positive semidefinite with kernel of dimension one. Matrices of finite type correspond to connected Dynkin diagrams, and matrices of affine type correspond to connected affine Dynkin diagrams. See V. Kac, *Infinite-dimensional Lie algebras*, CUP, for more details.

Remark. Given an affine Dynkin diagram $\widetilde{\Sigma}$, one can define an *affine Weyl group* of $\widetilde{\Sigma}$ as a Coxeter group defined by $\widetilde{\Sigma}$. This group has a nice geometric interpretation: it is a group generated by *affine* reflections in the Euclidean space $\mathbb{E}^n$, i.e. a group generated by reflections in hyperplanes which may not pass through the origin.

Example. Consider the affine Dynkin diagram $\widetilde{A}_2$, it is a cycle of length 3. The corresponding affine Weyl group is $\widetilde{W} = \langle s_1, s_2, s_3 \mid s_i^2, (s_is_j)^3 \rangle$. $\widetilde{W}$ acts on the plane as follows: s_1 and s_2 are usual reflections in lines $\alpha_1^\perp$ and $\alpha_2^\perp$ (where α_i are simple roots of A_2), and s_3 is a reflection with respect to the line $(v, \alpha_0) = -1$. The fundamental domain is a regular triangle.

Table 10.1: **Affine Dynkin diagrams.** Connected affine Dynkin diagrams. Every diagram contains $n+1$ nodes. The lowest root is marked.



In general, if an affine Dynkin diagram $\widetilde{\Sigma}$ is an extension of a Dynkin diagram Σ of a root system Δ, then the fundamental domain of $\widetilde{W}$ is a compact simplex bounded by n hyperplanes $\alpha_i^\perp$ (where α_i are simple roots of Δ), and a hyperplane $(v, \alpha_0) = -1$ (this simplex is called *alcove*). Group $\widetilde{W}$ is infinite and generated by reflections in the walls of the alcove.

Remark. If Δ is not simply-laced, one can see that the highest (and lowest) root is always long. However, the closure of the chamber C_0 also contains a short root. This leads to more extended Dynkin diagrams with the same affine Weyl group (the difference is in the lengths of roots only, i.e. in the directions of arrows in the extended Dynkin diagram).

11 Coxeter number

11.1 Coxeter element

Let (W, S) be an irreducible Coxeter system, where W is a finite Coxeter group of rank n, let $\{s_i = r_{\alpha_i}\}$ be simple reflections. We will assume by default that W is a Weyl group of a root system Δ with simple roots $\Pi = \{\alpha_i\}$, but everything in this section works for non-crystallographic groups as well.

Definition 11.1. An element $c = s_{i_1} \dots s_{i_n}$, where all indices are distinct, is called a *Coxeter element*.

In other words, a Coxeter element is a product of all simple reflections in some order. Clearly, there are $n!$ such elements, but some of them may coincide.

Example 11.2. If $\Delta = A_3$, then there are 6 Coxeter elements. However, as s_1 and s_3 commute, there are only four distinct ones: $s_1s_2s_3$, $s_3s_2s_1$, $s_2s_3s_1$ and $s_3s_1s_2$. Moreover, $s_1(s_1s_2s_3)s_1 = s_2s_3s_1 = s_2(s_3s_1s_2)s_2 = s_2(s_1s_3s_2)s_2 = s_2s_1(s_3s_2s_1)s_1s_2$, so all Coxeter elements in A_3 are conjugated to each other.

The same pattern can be seen in general.

Lemma 11.3. *All Coxeter elements are conjugated in W.*

Proof. First, observe that all cyclic shifts are conjugated: $s_{i_1}(s_{i_1}\dots s_{i_n})s_{i_1} = s_{i_2}\dots s_{i_n}s_{i_1}$ etc. Also, if two neighboring reflections commute, we may swap them without changing the element. We claim that using these two operations we can get *all* $n!$ Coxeter elements.

The proof is by induction on n. If $n = 1$ there is not much to prove, so assume the statement holds for $n-1$. Consider any Coxeter element c on letters $s_1, \dots, s_n$, we can assume s_n is a *leaf* of the Coxeter diagram of W, i.e. it commutes with all but one simple reflections. In particular, the Coxeter diagram of the subgroup $W_{S\setminus\{s_n\}}$ is connected, so we can apply the induction assumption.

By the induction assumption, any Coxeter element on letters $s_1, \dots, s_{n-1}$ can be taken to $s_1 \dots s_{n-1}$ by cyclic shifts and swapping commuting neighbors. Apply this procedure to c, we claim that the result is $s_1 \dots s_{n-1}$ with s_n inserted in one of n possible places. Indeed, s_n allows us to permute commuting s_i and s_j as s_n must commute with at least one of them, so we have either $s_is_ns_j = s_is_js_n = s_js_is_n$ or $s_is_ns_j = s_ns_is_j = s_ns_js_i$. Also, we can still perform cyclic shifts: if the word starts with $s_ns_i\dots$ and s_n does not commute with s_i, then we can do cyclic shift in s_n first and then in s_i.

Now, once we got $s_1 \dots s_{n-1}$ with s_n inserted in one of n possible places, we can carry s_n either to the start of the word of to the end (the only s_i not commuting with s_n lies on one side of it), and then perform a cyclic shift if needed.

$\square$

Corollary 11.4. *All Coxeter elements have the same order.*

Definition 11.5. The order of a Coxeter element is called the *Coxeter number* of Δ (or W) and is denoted by h.

Example 11.6.

- $\Delta = A_2$, $(s_1s_2)^3 = e$, so $h = 3$.
- $\Delta = A_3$, $(s_1s_2s_3)^h = e$, how to find h? One way is to play with words: observe that h must be even (e.g. by the Deletion Condition), so try $h = 2$. We get

$$(s_1s_2s_3)^2 = s_1s_2s_3s_1s_2s_3 = s_1s_2s_1\,s_3s_2s_3 = s_2s_1\,s_2s_3s_2s_3 = s_2s_1s_3s_2 = s_2(s_1s_3)s_2 \neq e,$$

  so $h \geq 4$. Now,

$$(s_1s_2s_3)^4 = (s_2(s_1s_3)s_2)^2 = s_2(s_1s_3)^2s_2 = e,$$

  so $h = 4$.

  Another way is to observe that for any A_{n-1} a Coxeter element $c = s_1 \dots s_{n-1}$ can be written as a product of transpositions:

$$c = s_1 \dots s_{n-1} = (12)(23)(34)\dots(n-1\,n) = (1\,2\,3\dots n-1\,n),$$

  so it is a cycle of length n, and thus $h(A_{n-1}) = n$.

  In general one can write matrices of the action of every simple reflection, take the product, and then compute eigenvalues.

Exercise. Compute $h(B_3)$.

11.2 Exponents

Observe that since all Coxeter elements are conjugated, they have the same characteristic polynomial, and thus the same eigenvalues. All eigenvalues are of the form ξ^m, where ξ is a primitive h-th root of unity, and $m \le h-1$.

Definition 11.7. The *exponents* $m_1 \le m_2 \le \dots \le m_n$ of W are all $m \in \mathbb{Z}_h$ such that ξ^m are eigenvalues of a Coxeter element. Exponents are counted with multiplicities.

Example 11.8. Let $\Delta = A_2$, then $c = s_1 s_2$ acts on $\mathbb{R}^2$ as a rotation by $2\pi/3$. Therefore, the eigenvalues of c are $\exp(\frac{2\pi i}{3})$ and $\exp(\frac{4\pi i}{3})$. Since $h = 3$, the primitive root of unity is $\xi = \exp(\frac{2\pi i}{3})$, so we get $m_1 = 1$ and $m_2 = 2$.

Exercise 11.9. Show that for $\Delta = A_n$ the exponents are $1, 2, \dots, n$.

Lemma 11.10. *$m_1 > 0$, i.e. 1 is not an eigenvalue of c.*

Proof. Let $c = s_1 \dots s_n$ and suppose there exists $v \in \mathbb{R}^n$ such that $c(v) = v$, $v \ne 0$. Then $s_1 \dots s_n v = v$ implies $s_2 \dots s_n v = s_1 v$. Let $s_i = r_{\alpha_i}$.

Note that $s_1 v = v + c_1\alpha_1$, and $s_2 \dots s_n v = v + \sum\limits_{i \ge 2} c_i \alpha_i$, so we obtain the equality $c_1\alpha_1 = \sum\limits_{i\ge 2} c_i\alpha_i$. However, simple roots are linearly independent, which implies that all $c_i = 0$. Therefore, $s_1 v = v$ and $s_2 \dots s_n v = v$. Applying the same argument $n-1$ times, we get $s_i v = v$ for every i, which implies that $(v, \alpha_i) = 0$ for every i. Since the dot product is not degenerate, this means that $v = 0$, so we came to a contradiction.

□

Lemma 11.11. *Let $m_1 \le \dots \le m_n$ be exponents of W. Then $\sum m_i = \frac{nh}{2}$.*

Proof. The characteristic polynomial of c has real coefficients, so every eigenvalue has a complex conjugate one. Observe that conjugate to ξ^{m_i} is $\xi^{-m_i} = \xi^{h-m_i}$, so if m_i is an exponent of multiplicity l then $h - m_i$ is also an exponent of the same multiplicity. Due to Lemma 11.10, the only real eigenvalue can be $-1 = \xi^{h/2}$. Therefore, if we denote the multiplicity of $h/2$ by k, then the sum of exponents is $k\frac{h}{2} + \frac{n-k}{2}h = \frac{nh}{2}$. □

11.3 Coxeter plane

We now want to explore the geometry of the action of the Coxeter element and derive some interesting corollaries.

Since the Dynkin diagram Σ of Δ is a tree, we can color nodes in black and white such that neighboring nodes are of different colors (in general, graphs with this properties are called *bipartite*). Index white nodes $1, \dots, k$ and black nodes $k+1, \dots, n$. Consider $c' = s_1 \dots s_k$ and $c'' = s_{k+1} \dots s_n$.

Definition 11.12. Coxeter element $c = c'c''$ is called *bipartite.*

From now on we assume c is bipartite – we do not loose any generality as all Coxeter elements are conjugate.

Recall that a *dual* basis $\{\omega_i\}$ to Π is defined by $(\alpha_i, \omega_j) = 1$ if $i = j$ and $(\alpha_i, \omega_j) = 0$ otherwise.

Exercise 11.13. Let $A = (a_{ij}) = (\alpha_i, \alpha_j)$. Show that $\alpha_j = \sum\limits_i a_{ij}\omega_i$.

Corollary 11.14. *In the basis $\{\omega_i\}$, $A\omega_j = \alpha_j$.*

Lemma 11.15. *Matrix $A = (\alpha_i, \alpha_j)$ has a positive eigenvalue λ with a positive eigenvector $(\lambda_1, \dots, \lambda_n)$, i.e. $\lambda_i > 0$ for all i.*

Lemma follows from Perron-Frobenius Theorem (see HW 9.3).

Theorem 11.16. *Let $(\lambda_1, \ldots, \lambda_n)$ be a positive eigenvector of $A = (\alpha_i, \alpha_j)$. Denote*

$$\mu = \sum_{i=1}^{k} \lambda_i \omega_i \quad \text{and} \quad \nu = \sum_{j=k+1}^{n} \lambda_j \omega_j.$$

Then

(1) *The plane H spanned by μ and ν is preserved (not pointwise) by c' and c'', and thus by c.*

(2) *c' and c'' act on H as reflections.*

(3) *The order of c restricted to H is h.*

H is called the Coxeter plane *of c.*

Proof. The first goal is to compute $(\lambda - 1)\mu$ and $(\lambda - 1)\nu$.

Since $(\lambda_1, \ldots, \lambda_n)$ is an eigenvector of A, we have

$$A(\sum_{i=1}^{n} \lambda_i \omega_i) = \lambda(\sum_{i=1}^{n} \lambda_i \omega_i) = \sum_{i=1}^{n} \lambda \lambda_i \omega_i,$$

and by Exercise 11.13 we have

$$A(\sum_{i=1}^{n} \lambda_i \omega_i) = \sum_{i=1}^{n} \lambda_i A \omega_i = \sum_{i=1}^{n} \lambda_i \alpha_i$$

Take the scalar product of the vector above with α_i, $i \leq k$. We get

$$(\sum_{m=1}^{n} \lambda_m \alpha_m, \alpha_i) = (\sum_{m=1}^{n} \lambda \lambda_m \omega_m, \alpha_i) = \lambda \lambda_i.$$

On the other hand,

$$(\sum_{m=1}^{n} \lambda_m \alpha_m, \alpha_i) = \sum_{m=1}^{n} \lambda_m (\alpha_m, \alpha_i) = \sum_{m=1}^{n} \lambda_m a_{im} = \lambda_i + \sum_{j=k+1}^{n} \lambda_j a_{ij}$$

as $a_{im} = 0$ for $m \leq k$ unless $m = i$. Thus, we see that $\lambda \lambda_i = \lambda_i + \sum\limits_{j=k+1}^{n} \lambda_j a_{ij}$, or, equivalently,

$$\lambda_i(\lambda - 1) = \sum_{j=k+1}^{n} \lambda_j a_{ij} \quad \text{for } i = 1, \ldots, k.$$

Therefore,

$$(\lambda - 1)\mu = (\lambda - 1)\sum_{i=1}^{k} \lambda_i \omega_i = \sum_{i=1}^{k} \lambda_i (\lambda - 1)\omega_i = \sum_{i=1}^{k} \left(\sum_{j=k+1}^{n} \lambda_j a_{ij} \right) \omega_i = \sum_{j=k+1}^{n} \lambda_j \left(\sum_{i=1}^{k} \alpha_{ij} \omega_i \right) =$$
$$= \sum_{j=k+1}^{n} \lambda_j \left(\sum_{m=1}^{n} a_{mj} \omega_i - \sum_{i=k+1}^{n} a_{ij} \omega_i \right) = \sum_{j=k+1}^{n} \lambda_j (\alpha_j - \omega_j) = \sum_{j=k+1}^{n} \lambda_j \alpha_j - \nu$$

since $a_{ij} = 0$ if both $i, j \geq k+1$ unless $i = j$. Similarly, we have $(\lambda - 1)\nu = \sum_{i=1}^{k} \lambda_i \alpha_i - \mu$. Summarizing, we get

$$\sum_{i=1}^{k} \lambda_i \alpha_i - (\lambda - 1)\nu = \mu \quad \text{and} \quad \sum_{j=k+1}^{n} \lambda_j \alpha_j - (\lambda - 1)\mu = \nu.$$

We will now use the obtained formulae to look how c' and c'' act on μ and ν. We have

$$c'(\nu) = r_{\alpha_1} \dots r_{\alpha_k} \left(\sum_{j=k+1}^{n} \lambda_j \omega_j \right) = \sum_{j=k+1}^{n} \lambda_j \omega_j = \nu$$

as $(\alpha_i, \omega_j) = 0$ for $i \leq k < j$. Similarly, $c''(\mu) = \mu$. Further,

$$c'(\mu) = c' \left(\sum_{i=1}^{k} \lambda_i \alpha_i - (\lambda - 1)\nu \right) = - \left(\sum_{i=1}^{k} \lambda_i \alpha_i \right) - (\lambda-1)\nu = -(\mu + (\lambda-1)\nu) - (\lambda-1)\nu = -\mu - 2(\lambda-1)\nu$$

since $c'(\alpha_i) = -\alpha_i$ for $i \leq k$. Similarly, $c'(\nu) = -\nu - 2(\lambda - 1)\mu$. Therefore, both c' and c'' take μ and ν to their linear combinations, so both c' and c'' preserve H.

Furthermore, restriction of c' on H is a non-trivial element of $O_2(\mathbb{R})$ with eigenvalue 1 (as $c'(\mu) = \mu$), so the other eigenvalue must be -1, and this c' acts on H as a reflection. Similarly, the same is true for c''. In particular, the restriction of c on H is a rotation.

Therefore, we have proved (1) and (2). To prove (3), observe first that every ω_i is a 1-dimensional face of the initial chamber C_0 (cf. the proof of Lemma 9.13), and thus $\mu + \nu = \sum_{i=1}^{n} \lambda_i \omega_i \in C$ as all $\lambda_i > 0$. In particular, we see that $H \cap C \neq \emptyset$. Take any $p \in H \cap C$. If $c^m(p) = p$, then $c^m = e$ by Theorem 7.7, which implies that the order of c restricted to H is precisely h, so the proof is complete.

□

Lemma 11.17. *Let $m_1 \leq \dots \leq m_n$ be the exponents, let h be the Coxeter number. Then*

(1) $m_1 = 1$, $m_n = h - 1$;

(2) $|\Delta^+| = \frac{nh}{2}$.

Proof. Observe that any positive linear combination $a\mu + b\nu$ belongs to C_0, i.e., the angle in H formed by rays $\mathbb{R}_+\mu$ and $\mathbb{R}_+\nu$ lies in C_0. This implies that this angle is not intersected by any mirror of W. Also, the reflections (on H) c' and c'' in $\mathbb{R}\nu$ and $\mathbb{R}\mu$ respectively generate a dihedral group of order $2h$. We want to show that the angle formed by rays $\mathbb{R}_+\mu$ and $\mathbb{R}_+\nu$ is equal π/h – this will imply that $m_1 = 1$.

Suppose that $\angle(\mathbb{R}_+\mu, \mathbb{R}_+\nu) = \frac{m\pi}{h}$, where $m \geq 2$. Then the intersection $H \cap C_0$ is an angle of size at least $2\pi/h$. The orbit of $H \cap C_0$ under the group generated by c' and c'' consists of $2h$ copies of it, and no two copies can overlap (otherwise they would need to coincide as each angle belongs to one chamber only, which, in its turn, would mean that the corresponding elements of the dihedral group must be equal, which cannot happen). This immediately leads to a contradiction.

Therefore, $m_1 = 1$, and thus the complex conjugate eigenvalue gives $m_n = h - 1$, so (1) is proved.

To prove (2), observe first that mirrors of W can intersect H along copies of $\mathbb{R}\mu$ and $\mathbb{R}\nu$ under the dihedral group. There are $2h$ angles formed by these lines, so there are h lines in total. Our plan is to count the number of mirrors intersecting H along $\mathbb{R}\mu$ and $\mathbb{R}\nu$.

Denote $H_i = \alpha_i^\perp$. By definition, $\mathbb{R}\mu = (H_{k+1} \cap \dots \cap H_n) \cap H$. We want to prove the following claim: no other mirror of W contains $\mathbb{R}\mu$.

Suppose there is some $H_\alpha \cap H = \mathbb{R}\mu$, $\alpha = \sum_i b_i\alpha_i$, $b_i \geq 0$. This implies $(\alpha, \mu) = 0$, so we have

$$0 = (\alpha, \mu) = \left(\sum_{i=1}^{n} b_i\alpha_i, \sum_{i=1}^{k} \lambda_i\omega_i\right) = \sum_{i=1}^{k} b_i\lambda_i.$$

As all $\lambda_i > 0, b_i \geq 0$, this implies that $b_i = 0$ for $i = 1, \dots, k$, so $\alpha = \sum_{i=k+1}^{n} b_i\alpha_i$. However, all α_i for $i = k+1, \dots, n$ are mutually orthogonal, and thus by HW 18.2, $\alpha = \alpha_i$ for some $i > k$, so the claim is proved.

Therefore, $\mathbb{R}\mu$ is contained in $n-k$ mirrors of W. Similarly, $\mathbb{R}\nu$ is contained in k mirrors of W. The copies of $\mathbb{R}\mu$ and $\mathbb{R}\nu$ under the action of the dihedral group generated by c' and c'' are also contained in $n-k$ and k mirrors respectively. If h is even, then there are $h/2$ lines of each of the two types, so $|\Delta^+| = \frac{h}{2}(n-k) + \frac{h}{2}k = \frac{nh}{2}$. If h is odd, then $\mathbb{R}\mu$ and $\mathbb{R}\nu$ are equivalent under the action of the dihedral group, and thus $k = n-k$, i.e. $k = n/2$. Therefore, $|\Delta^+| = hn/2$ as well.

□

Combining Lemmas 11.11 and 11.17, we get the following corollary.

Corollary 11.18. *The number of positive roots in a root system is equal to the sum of exponents.*

Exercise. If h is even, then the longest element g_0 of W is precisely $c^{h/2}$.

11.4 More about Coxeter number and exponents

We list here more facts without proofs.

Theorem 11.19. *Let $m_1 \leq \dots \leq m_n$ be the exponents, define partition $\mu = (m_n, \dots, m_1) \vdash |\Delta^+|$. Let $\lambda = (l_1, \dots, l_{m_n})$ be the dual Young diagram. For $\alpha = \sum c_j\alpha_j \in \Delta^+$ let the height $\operatorname{ht}\alpha$ denote $\sum c_j$. Then $l_i = \#\{\alpha \in \Delta^+ \mid \operatorname{ht}\alpha = i\}$.*

Example 11.20. Consider $\Delta = A_n$. We know the exponents are $m_i = i$. This implies $l_i = n+1-i$. Indeed, roots of height i are precisely $e_j - e_{j+i} = \alpha_j + \dots + \alpha_{j+i-1}$, so $j = 1, \dots, n+1-i$, and thus the number of roots of height i is precisely $n+1-i$.

Theorem 11.21. *The order of the Weyl group can be computed as follows:*

$$|W| = (1+m_1)(1+m_2)\dots(1+m_n)$$

The proof (following R. Steinberg, *Finite reflection groups*, Trans. Amer. Math. Soc. 91 (1959)) of the following theorem is contained in HW 19.3.

Theorem 11.22. *Let $\widetilde{\alpha}_o$ be the highest root. Then $\operatorname{ht}\widetilde{\alpha}_0 + 1 = h$.*

The next two sections are NON-EXAMINABLE.

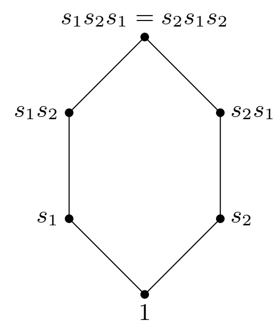
12 Bruhat order

12.1 Bruhat order and weak Bruhat order

Let (G, S) be a Coxeter system.

Definition 12.1 (Weak Bruhat order)**.** We say that $g_1 \lessdot_R g_2$ if $g_2 = g_1 s_i$ and $l(g_2) = l(g_1) + 1$, where $s_i \in S$. Thus, we can define $g_1 \leq_R g_2$ if $g_2 = g_1 s_{i_1} \dots s_{i_k}$ and $l(g_2) = l(g_1) + k$. The partial order $\leq_R$ is called a *right weak Bruhat order*. Similarly, we can define *left weak Bruhat order* as $g_1 \leq_L g_2$ if $g_2 = s_{i_k} \dots s_{i_1} g_1$ and $l(g_2) = l(g_1) + k$.

Example 12.2. Let $G = S_3$. The Hasse diagram of weak right order is shown below.

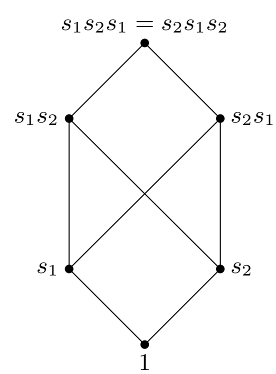


Remark. Right and left weak orders are isomorphic via an (anti-)isomorphism of G taking $g \mapsto g^{-1}$.

Definition 12.3 (Bruhat order)**.** Recall that R is the set of reflections of G. We say that $g_1 \leq g_2$ if $g_2 = g_1 t_1 \dots t_k$ and $l(g_1 t_1 \dots t_{i+1}) > l(g_1 t_1 \dots t_i)$, where $t_i \in R$. The partial order $\leq$ is called the *Bruhat order*.

Exercise 12.4. Define a partial order on G by $g_1 \leq g_2$ if $g_2 = t_k \dots t_1 g_1$ and $l(t_{i+1} \dots t_1 g_1) > l(t_i \dots t_1 g_1)$, where $t_i \in R$. Show that this partial order coincides with Bruhat order.

Example 12.5. Let $G = S_3$. The Hasse diagram of Bruhat order is shown below.



Remark. It is easy to see why the weak order is indeed called *weak*: it has less relations than Bruhat order, but if $g_1 \leq_R g_2$ or $g_1 \leq_L g_2$ then $g_1 \leq g_2$.

Example 12.6. In the previous example $g_1 \leq g_2$ if $l(g_1) < l(g_2)$. This is not usually the case. For example, in S_4, s_1 is incomparable with $s_2 s_3$. Indeed, since $l(s_2 s_3) = l(s_1) + 1$, $s_2 s_3$ being comparable to s_1 would imply that $s_1(s_2 s_3)$ is a reflection, which is not the case.

12.2 Inversions

We will assume by default that W is a Weyl group of a root system Δ with simple roots $\Pi = \{\alpha_i\}$, but everything in this section works for non-crystallographic groups as well.

Definition 12.7. Let W be a Weyl group of a root system Δ, $\Pi = \{\alpha_i\}$ are simple roots. Given $w \in W$, $\alpha \in \Delta^+$ is an *inversion* of w if $w(\alpha) \in \Delta^-$. Denote by $\mathrm{Inv}\,(w)$ the set of inversions of w.

Example. Let $\Delta = A_2$, $w = s_1 s_2$. There are three positive roots, and $w(\alpha_1) = \alpha_2$, $w(\alpha_2) = -\alpha_1 - \alpha_2$, and $w(\alpha_1 + \alpha_2) = -\alpha_1$. Therefore, $\mathrm{Inv}\,(w) = \{\alpha_2, \alpha_1 + \alpha_2\}$. In particular, we see that $|\mathrm{Inv}\,(w)| = l(w)$.

Lemma 12.8. *Let W be a Weyl group. Then $|\mathrm{Inv}\,(w)| = l(w)$ for any $w \in W$.*

Proof. We know that the length of w is equal to the number of reflections whose mirrors separate the initial Weyl chamber C_0 from wC_0 (see Remark 8.11). Take any $v \in C_0$ and $\alpha \in \Delta^+$, then the mirror of r_α separates C_0 from wC_0 if and only if $(\alpha, w(v)) < 0$, which is equivalent to $(w^{-1}(\alpha), v) < 0$, i.e. $\alpha \in \mathrm{Inv}\,(w^{-1})$. Therefore, $l(w) = |\mathrm{Inv}\,(w^{-1})|$. Since $l(w) = l(w^{-1})$, we get $|\mathrm{Inv}\,(w^{-1})| = l(w^{-1})$ for any $w \in W$, which implies the statement of the lemma. □

Remark. Another proof of the lemma can be found in HW 16.1.

As a corollary of the proof, we obtain the following statement.

Corollary 12.9. $\mathrm{Inv}\,(w) = \{\alpha \in \Delta^+ \mid \alpha \in R(w^{-1})\}$.

Recall that for a symmetric group we had another definition of inversion (see Def. 3.2): given a permutation $w = w_1 \dots w_n$, an inversion is a pair $i < j$ such that $w_i > w_j$.

Example 12.10. Let $W = S_4$, i.e. $\Delta = A_3$. Take $w = s_1 s_2 s_3 = 2\,3\,4\,1$. Then $w^{-1} = 4\,1\,2\,3$, and the inversions of w^{-1} are $(1,2)$, $(1,3)$ and $(1,4)$. Now, the R-sequence of w is $\{s_1 = (12), s_1 s_2 s_1 = (12)(23)(12) = (13), s_1 s_2 s_3 s_2 s_1 = (12)(23)(34)(23)(12) = (12)(24)(12) = (14)\}$, so we see that the inversion set of w^{-1} coincides with the R-sequence of w. In other words, for w the two definitions of inversion coincide.

Theorem 12.11. *The two definitions of inversions for symmetric group coincide.*

Proof. Let $w = w_1 w_2 \dots w_n \in S_n$. An inversion of w^{-1} is a pair (w_i, w_j) such that $i < j$ and $w_i > w_j$. The symmetric group $W = S_n$ acts on $\mathbb{R}^n$ by permutation of coordinates, i.e. w maps $e_i \mapsto e_{w_i}$. If $x \in \mathbb{R}^n$, then $w(x)_{w_i} = x_i$. The Weyl chamber C_0 is given by equations $x_1 > x_2 > \dots > x_n$.

Let $x \in C_0$. If $i < j$ and $w_i > w_j$, then $x_i < x_j$ and $x_{w_i} > x_{w_j}$, which is equivalent to $w(x)_{w_i} = x_i > x_j = w(x)_{w_j}$, i.e. $w(x)_{w_i} > w(x)_{w_j}$, while $w_i > w_j$. This is equivalent to the hyperplane $x_{w_i} = x_{w_j}$ separating C_0 from wC_0, i.e. to $(w_i, w_j) \in R(w)$, and thus to $(w_i, w_j) \in I(w^{-1})$ due to Cor. 12.8. □

Remark. Theorem 12.11 shows that Inv can be considered as a generalization of the inversion statistics to all Weyl groups (and, in fact, to all finite Coxeter groups).

We list below some corollaries.

Corollary 12.9 can be reformulated as follows.

Corollary 12.12. *Let $w = s_1 \dots s_k$ be a reduced expression, $w = r_k \dots r_1$, where $\{r_1 = r_{\alpha_1}, \dots, r_k = r_{\alpha_k}\}$ is R-sequence of w. Then $\{\alpha_i\} = \mathrm{Inv}\,(w^{-1})$.*

Assume $u \leq_R w$, i.e. $w = us_1 \dots s_m$. Then clearly $R(u) \subset R(w)$, so $\mathrm{Inv}\,(u^{-1}) \subset \mathrm{Inv}\,(w^{-1})$. In fact, the converse is also true.

Lemma 12.13. *Let $u, w \in W$. Then $u \leq_R w$ if and only if $R(u) \subset R(w)$.*

Proof. Let $l(u) = k$, $u = s_1 \dots s_k = r_k \dots r_1$, and assume $R(u) \subset R(w)$. We claim that for any $i \le k$ w has a reduced expression $s_1 \dots s_i\, s'_{i+1} \dots s'_m$, this will immediately imply the lemma.

The proof is by induction on i, the case $i = 0$ is evident. Suppose the claim holds for i, i.e. $w = s_1 \dots s_i\, s'_{i+1} \dots s'_m$. Then

$$r_{i+1} = s_1 \dots s_i s_{i+1} s_i \dots s_1 = s_1 \dots s_i s'_{i+1} \dots s'_l \dots s'_{i+1} s_i \dots s_1$$

for some l between $i+1$ and m: indeed, $r_{i+1} \in R(u) \subset R(w)$, and since u is reduced $r_{i+1} \neq r_q$ for $q \neq i+1$ and thus $l > i$. Therefore, w has expression

$$\begin{aligned}w = r_{i+1}^2 w = (s_1 \dots s_i s_{i+1} s_i \dots s_1)(s_1 \dots s_i s'_{i+1} \dots s'_l \dots s'_{i+1} s_i \dots s_1) s_1 \dots s_i\, s'_{i+1} \dots s'_m = \\ (s_1 \dots s_i s_{i+1})(s_i \dots s_1)(s_1 \dots s_i)(s'_{i+1} \dots s'_{l-1})(s'_l \dots s'_{i+1} s_i \dots s_1)(s_1 \dots s_i\, s'_{i+1} \dots s'_l)(s'_{l+1} \dots s'_m) = \\ (s_1 \dots s_i s_{i+1})(s'_{i+1} \dots s'_{l-1})(s'_{l+1} \dots s'_m)\end{aligned}$$

which starts with $s_1 \dots s_i s_{i+1}$ and consists of precisely m letters, and thus is reduced.

□

Corollary 12.14. *Let R be the set of reflections of W, denote by $P(R)$ the set of all subsets of R. The map $W \to P(R)$ taking $w \in W$ to $R(w)$ is an order-preserving embedding of posets.*

13 Coxeter-Catalan combinatorics

Given a Coxeter system (W, S), define a number

$$N(W) = \prod_{i=1}^{n} \frac{m_i + h + 1}{m_i + 1},$$

where h is the Coxeter number, m_i are exponents, and W is the rank. We also write $N(W) = N(\Delta)$ if W is the Weyl group of root system Δ.

Example 13.1. Let $W = S_{n+1}$. Then $h = n+1$, $m_i = 1, \dots, n$, so

$$N(S_{n+1}) = \frac{(1+(n+2))(2+(n+2))\dots(n+(n+2))}{(1+1)(2+1)\dots(n+1)} = \frac{(2n+2)!}{(n+2)!(n+1)!} = \frac{1}{n+2}\binom{n+2}{n+1} = C_{n+1}$$

Remark 13.2. $N(W)$ can be understood as a generalization of the Catalan numbers to other Weyl groups. In other words, objects counted by Catalan numbers are of "A_n type", and they can often be generalized to other root systems/Weyl groups. We consider some examples below.

13.1 Non-nesting partitions

Lemma 13.3. *There is a bijection between non-nesting partitions of $[n+1]$ and antichains in the root poset of A_n.*

Proof. The map from the set of antichains to the set of partitions is constructed as follows: given an antichain, draw an arc (i, j) for every root $e_i - e_j$ it contains, and define a partition by this arc diagram. The resulting partition is indeed non-nesting: if $i < j < k < l$, then $e_i - e_l > e_j - e_k$, so nesting arcs cannot appear. The map is injective by construction. Also, any non-nesting partition can be obtained: take any admissible arc diagram, then any two arcs define incomparable roots.

□

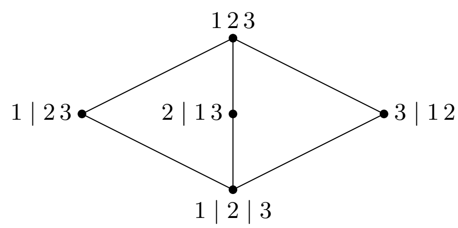
Therefore, we can consider antichains in the root poset of a root system Δ as a generalization of non-nesting partitions.

Theorem 13.4 (Athanasiadis, Postnikov, Reiner)**.** *The number of antichains in the root poset of a root system Δ is $N(\Delta)$.*

13.2 Non-crossing partitions NC_n

Define a partial order on the set NC_n of non-crossing partitions of $[n]$: we say $P_1 \leq P_2$ if every block of P_1 is contained in some block of P_2.

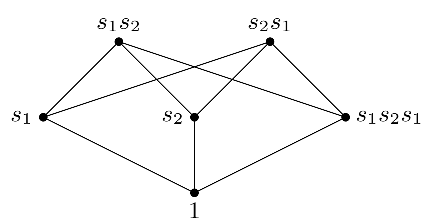
Example. Let $n = 3$. Then the Hasse diagram of the order is



The poset NC_n with the order as above is called an *NC_n-lattice* (check it is indeed a lattice!).

Now consider a Weyl group W, and define an *absolute order* on W as follows. Let R be the set of reflections, and for $w \in W$ let $L(w)$ denote the "R-length" of w, i.e. the minimal k such that $w = r_1 \dots r_k$, $r_i \in R$. We say $w_1 \preceq w_2$ if $w_2 = w_1 w$ and $L(w_2) = L(w_1) + L(w)$.

Example. Let $\Delta = A_n$, $n = 2$. The the Hasse diagram of the absolute order looks as follows.



Definition 13.5. Let P be a poset with partial order $\leq$. Given $p_1, p_2 \in P$, an *interval* $[p_1, p_2]$ is defined by

$$[p_1, p_2] = \{p \in P \mid p_1 \leq p \leq p_2\}$$

Lemma 13.6. *Let $W = S_n$, c is a Coxeter element. Then the interval $[1, c]$ in the absolute order is isomorphic to the NC_n-lattice.*

Theorem 13.7 (Reiner, Bessis)**.** *The number of elements in $[1, c]$ is equal to $N(\Delta)$.*

Therefore, the interval $[1, c]$ in the absolute order of any Weyl group W can be considered as a generalization of the NC_n-lattice.

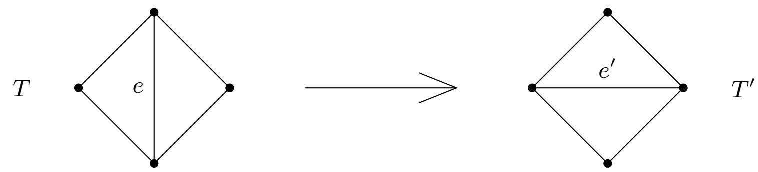
There is a more geometric way to interpret non-crossing partitions for Weyl groups which is due to Reading. Let $w \in [1, c]$, $w = r_1 \dots r_k$ reduced in the R-alphabet. Take the group Γ generated by $r_1, \dots, r_k$ – this is a parabolic subgroup (cf HW 17.2; note: this statement requires a proof). Then the absolute order is equivalent to containment of the corresponding parabolic subgroups. Take the smallest orbit O of W (e.g., for $\Delta = A_n$ the largest stabilizer is A_{n-1}, so the smallest orbit O has $n+1$ elements). Project the orbit to the Coxeter plane (defined by c) orthogonally (e.g., for A_n we get a regular $(n+1)$-gon). Then O is split into orbits of Γ, every orbit is precisely a block of the partition corresponding to w.

Details of the construction can be found in N. Reading, *Noncrossing partitions, clusters and the Coxeter plane*, Sém. Lothar. Combin. 63 (2010), Art. B63b.

13.3 Associahedra

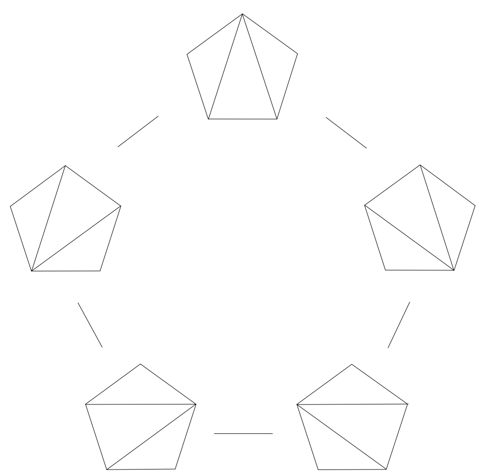
Recall: the number of triangulations of an $(n+2)$-gon is C_n.

Definition 13.8. Given a triangulation T and an edge e of T, a *flip* of T in e produces a new triangulation T' as follows: it substitutes e with the other diagonal of the corresponding quadrilateral, see the picture below.



An *exchange graph* of triangulations of an $(n+3)$-gon has all triangulations as its vertices, and two vertices are connected by an edge if the corresponding triangulations are related by a flip.

Example. Let $n = 2$, then there are five triangulations of a pentagon.



For $n = 3$, there are 14 triangulations of a hexagon, see Fig. 13.1.

The exchange graph is a 1-skeleton of an n-dimensional polytope which is called an *associahedron* (or Stasheff polytope).

One relation of the associahedron to the A_n root system can be seen from Theorem 13.10.

Definition 13.9. Let $W = S_{n+1}$, let C be a Weyl chamber of root system A_n, and let $x \in C$. The convex hull of the orbit Wx is a convex n-polytope called *permutohedron.*

Theorem 13.10 (Tonks)**.** *An n-dimensional associahedron can be obtained from an n-dimensional permutohedron by contraction of some edges.*

We will now make the relation of the associahedron to root system A_n more precise.

Definition 13.11. Let Δ be a root system, $\Pi = \{\alpha_i\}$ are simple roots. The set of *almost positive roots* $\Delta_{\geq -1}$ is defined by

$$\Delta_{\geq -1} = \Delta \sqcup \{-\alpha_i \mid \alpha_i \in \Pi\}$$

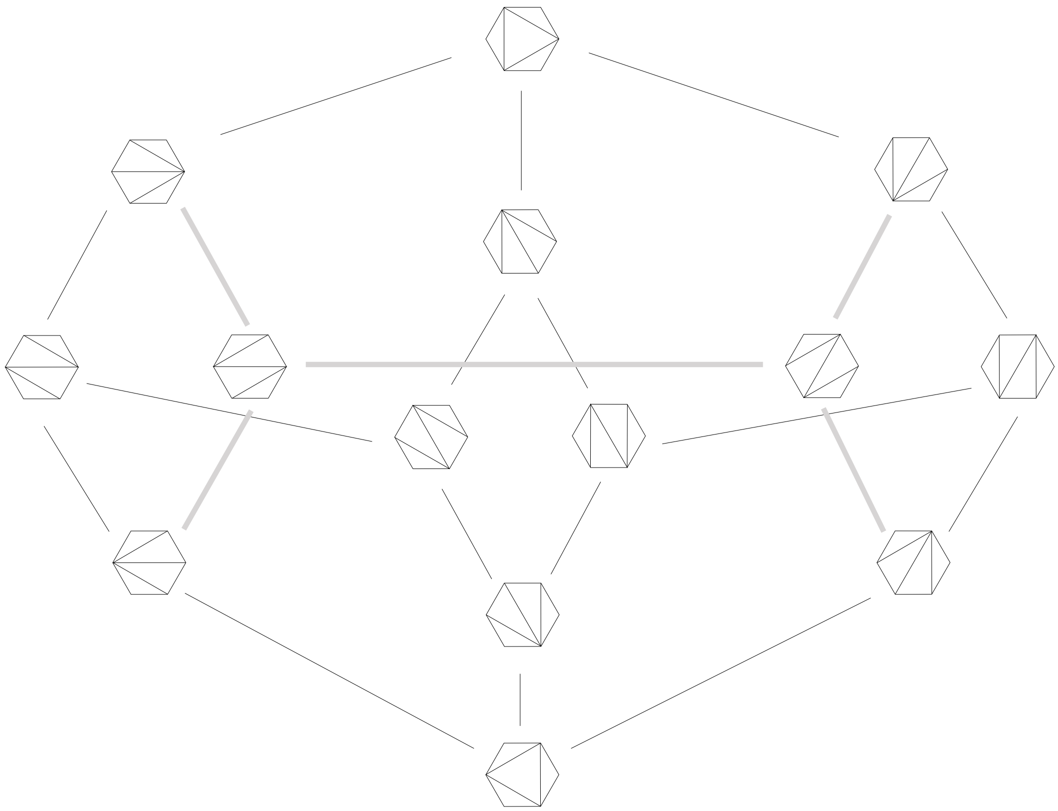


Figure 13.1: Exchange graph of triangulations of a regular hexagon.

Definition 13.12. Let $c = c'c''$ be a bipartite Coxeter element of a root system Δ, where $c' = \prod_{i=1}^{k} s_i$ and $c'' = \prod_{i=k+1}^{n} s_i$. Define $\tau', \tau'' : \Delta_{\geq -1} \to \Delta_{\geq -1}$ by

$$\tau'(\alpha) = \begin{cases} \alpha, & \alpha = -\alpha_i,\ i \geq k+1 \\ c'(\alpha) & \text{otherwise} \end{cases} \qquad \text{and} \qquad \tau''(\alpha) = \begin{cases} \alpha, & \alpha = -\alpha_i,\ i \leq k \\ c''(\alpha) & \text{otherwise} \end{cases}$$

Exercise. Check that τ', τ'' are involutions.

Example 13.13. Let $\Delta = A_2$, $c' = s_1 = r_{\alpha_1}$, $c'' = s_2 = r_{\alpha_2}$. Then τ' and τ'' act on $\Delta_{\geq -1}$ as follows.

$$\overset{\tau''}{\curvearrowleft}\!\!-\alpha_1 \xleftrightarrow{\tau'} \alpha_1 \xleftrightarrow{\tau''} \alpha_1 + \alpha_2 \xleftrightarrow{\tau'} \alpha_2 \xleftrightarrow{\tau''} -\alpha_2\overset{\tau'}{\curvearrowright}$$

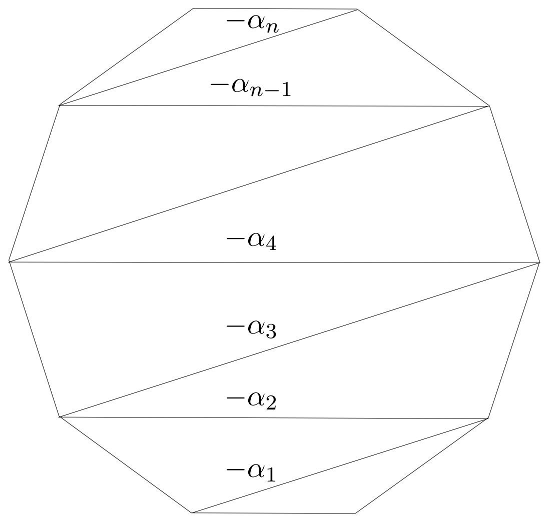
Theorem 13.14 (Fomin-Zelevinsky)**.** *There is a unique binary relation (called* compatibility*) on $\Delta_{\geq -1}$ satisfying the following two properties:*

(a) It is invariant with respect to the action of τ' and τ'', i.e. α is compatible with β if and only if $\tau'(\alpha)$ is compatible with $\tau'(\beta)$ if and only if $\tau''(\alpha)$ is compatible with $\tau''(\beta)$;

(b) $-\alpha_i$ is compatible with β if and only if the expression of β as a linear combination of α_j does not contain α_i.

Example 13.15. In A_2, there are five pairs of compatible elements: $(-\alpha_1, -\alpha_2), (-\alpha_1, \alpha_2), (-\alpha_2, \alpha_1), (\alpha_1, \alpha_1 + \alpha_2), (\alpha_2, \alpha_1 + \alpha_2)$.

Example 13.16. Let $\Delta = A_n$. Observe that the number of elements in $\Delta_{\geq -1}$ is $\frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}$, which is equal to the number of diagonals of $(n+3)$-gon. Consider a "staircase" triangulation of an $(n+3)-gon$ (see below), and assign to its edges negative simple roots as follows.



For every other diagonal d not in the triangulation, assign to it a root $\alpha_i + \cdots + \alpha_j$ if d intersect precisely edges of the triangulation with assigned roots $-\alpha_i, \ldots, -\alpha_j$. Then the roots are compatible if and only if the corresponding diagonals do not intersect. In particular, all triangulations are precisely maximal sets of mutually compatible roots.

This implies that the associahedron can be considered as follows: the vertices are maximal compatible sets of roots, and faces of dimension $n-k$ are collections of k compatible roots. For example, the 2-dimensional faces of 3-dimensional associahedron shown in Figure 13.1 correspond to roots from $\Delta_{\geq -1}$, or, equivalently, to diagonals: given a face, there is a unique diagonal which is present in every triangulation corresponding to a vertex of that face; in Figure 13.1, quadrilateral faces correspond to long diagonals of the hexagon, and pentagonal faces correspond to short diagonals.

Theorem 13.17 (Fomin-Zelevinsky)**.** *For every root system Δ of rank n, all maximal compatible sets in $\Delta_{\geq -1}$ have cardinality n. For every maximal compatible set Σ and any $\alpha \in \Sigma$ there exists a unique $\beta \notin \Sigma$ such that $(\Sigma \setminus \alpha) \cup \{\beta\}$ is maximal compatible. The number of maximal compatible sets is equal to $N(\Delta)$.*

Theorem 13.17 allows us to define an exchange graph of maximal compatible sets for any Weyl group. These are 1-skeletons of convex polytopes called *generalized associahedra*. Faces of dimension $n-k$ are precisely collections of k compatible roots. For details, see the lecture notes *Root systems and generalized associahedra* by Fomin and Reading, and references therein.