

Topics in Combinatorics IV, Homework 11 (Week 11)

Due date for starred problems: **Monday, February 02, 12pm.**

Let P be a cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$. A *symmetry* of P is a $g \in O_3(\mathbb{R})$ taking P to itself.

- 11.1.** (a) Show that symmetries of P compose a group, denote it by $\text{Sym } P$.
(b) Show that $\text{Sym } P$ acts on the set of faces of P transitively.
(c) Show that $\text{Sym } P$ acts transitively on the set of triples (v, e, f) , where v is a vertex of P , e is an edge, f is a face, and $v \in e \subset f$.
(d) An element $g \in O_3(\mathbb{R})$ is *orientation-preserving* if $\det g = 1$. Show that the subgroup $\text{Sym}^+ P$ of $\text{Sym } P$ consisting of all orientation-preserving symmetries of P is isomorphic to S_4 ; what does it permute?
(e) Compute the order of $\text{Sym } P$.
- 11.2.** (a) Show that $\text{Sym } P$ is generated by reflections. How many of them do you need to generate $\text{Sym } P$?
(b) Show that $\text{Sym } P$ cannot be generated by two reflections.

Let v be a vertex of P , $e \ni v$ be an edge of P , and $f \supset e$ be a face of P . Let $p_1 = v$, denote by p_2 the center of e , by p_3 the center of f , and by O the center of P (i.e., the origin of $\mathbb{R}^3$). Let C be the cone over triangle $p_1 p_2 p_3$ with apex O .

- 11.3.** (★) Show that three reflections in the walls of C generate $\text{Sym } P$. Write down the relations among these generators (i.e., give a presentation of $\text{Sym } P$ by generators and relations, where generators are the three reflections above).

Let G be a group acting on a set X . Recall that the *stabilizer* $\text{Stab}_G(x)$ of $x \in X$ in G is the set of elements of G fixing x , i.e. $\text{Stab}_G(x) = \{g \in G \mid gx = x\}$. For a set $U \subset X$ the stabilizer $\text{Stab}_G(U)$ is defined as the intersection of stabilizers of all points of U .

- 11.4.** Show that for every point $p \in \mathbb{R}^n$ the stabilizer $\text{Stab}_{\text{Sym } P}(p)$ is generated by all reflections $r \in \text{Sym } P$ such that $rp = p$.