Durham University
Pavel Tumarkin

Easter 2026

Topics in Combinatorics IV, Revision problems (Week 21)

These are examples from the second revision lecture. All HW problems are also revision problems.

R.1. Let $W = \langle s_1, \dots, s_4 \mid s_i^2, (s_2 s_j)^3 \text{ for } j \neq 2, (s_k s_l)^2 \text{ for } k, l \neq 2 \rangle$ be the Weyl group of type D_4. Show that the subgroup Γ of W generated by s_2 and $s_1 s_3 s_4$ is isomorphic to the dihedral group of type $G_2 = I_2(6)$.

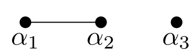
Denote $a = s_2$, $b = s_1 s_3 s_4$. Then Γ is generated by a, b with some relations. Observe that $a^2 = b^2 = 1$, so all elements of Γ are alternating products of a and b. A words with an odd number of letters cannot be trivial (as it is conjugated to a or b), so the only missing relation is $(ab)^m$ for some m. Since $ab = s_2 s_1 s_3 s_4$ is a Coxeter element of W, its order is the Coxeter number of D_4, which can be easily found from the formula $N = nh/2$, where N is the number of positive roots, h is the Coxeter number, n is the dimension. Namely, $n = 4$, $N = 12$ as positive roots of D_4 are of the type $e_i \pm e_j$ (where $1 \leq i < j \leq 4$), so $h = 6$. Therefore,

$$\Gamma = \langle a, b \mid a^2, b^2, (ab)^6 \rangle$$

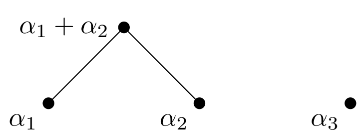
R.2. Draw Hasse diagrams of root poset P of root system $A_2 \oplus A_1$. Draw the Hasse diagram of the poset of order ideals $J(P)$.

Using explicit form of roots of the root systems, one obtains the following Hasse diagrams.

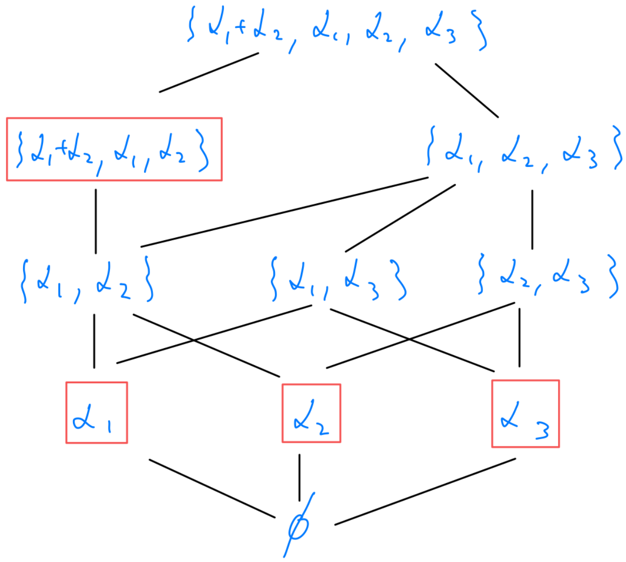
Disconnected Dynkin diagram looks as follows:



There are four positive roots of $A_2 \oplus A_1$: simple roots $\alpha_1, \alpha_2, \alpha_3$, and also the only non-simple positive root $\alpha_1 + \alpha_2$ of A_2. The Hasse diagram of the root poset is



The poset $J(P)$ is ranked, where the rank is the number of elements in the order ideal. Rank varies from 0 to 4, so the Hasse diagram is



Join-irreducible elements are those covering precisely one other element, they are boxed.

R.3. Let Δ be the root system of type D_5. Compute the Coxeter number of Δ. Find the exponents of the Weyl group of Δ.

The set of roots of Δ is $\{\pm e_i \pm e_j\}$, where $i, j = 1, \dots, 5$, $i < j$, and $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^5$.

The number of positive roots of D_5 is $N = 20$, the rank is $n = 5$. Using the formula $h = 2N/n$ we see that $h = 8$.

Now, we can use the result from lectures stating that all integers from 1 to $h - 1$ coprime with h are exponents. This implies that $1, 3, 5, 7$ are exponents. The sum of exponents is equal to $N = 20$, so the only missing exponent is 4. (Note that the last exponent can also be found as $h/2$ as it should provide the only real eigenvalue of the Coxeter element).

Alternatively, a linear map for a Coxeter element can be written explicitly. For example, for $c = r_{e_1-e_2} r_{e_2-e_3} r_{e_3-e_4} r_{e_4-e_5} r_{e_4+e_5}$ the matrix of c in the basis $\{e_i\}$ is

$$c = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

and it is easy to see that its order is 8.

The explicit expression for the matrix of a Coxeter element (see the alternative solution above) implies that the characteristic polynomial is $(-x - 1)(x^4 + 1)$, so the eigenvalues are $-1 = e^{\frac{2\pi i}{8} 4}$ and $e^{\frac{\pi i}{4} + \frac{2k\pi i}{4}} = e^{(1+2k)\frac{2\pi i}{8}}$, where $k = 0, 1, 2, 3$. Therefore, exponents are $1, 3, 4, 5, 7$.

Alternatively, one can use a result from lectures that the Young diagram $\lambda = (l_1, \dots, l_k)$ which is conjugate to the Young diagram $\mu = (m_5, m_4, m_3, m_2, m_1)$ (where m_i are the exponents) satisfies the following: l_i is equal to the number of positive roots of height i. Thus, we are left to compute the number of positive roots of every height.

A root of type $e_i - e_j$ can be written as

$$e_i - e_j = \sum_{k=i}^{j-1}(e_k - e_{k+1}),$$

so the height of $e_i - e_j$ is $j - i$. This number takes value 1 four times, 2 three times, 3 two times, and 4 one time.

A root of type $e_i + e_j$ can be written as

$$e_i + e_j = (e_i - e_5) + (e_j + e_5) = (e_i - e_5) + (e_j - e_4) + (e_4 + e_5),$$

so the height of $e_i + e_j$ is $(5-i)+(4-j)+1 = 10-(i+j)$. Note that if $j = 5$ then the previous calculation should be adjusted, but the answer is still correct. The number $10-(i+j)$ takes values $1, 2, 6, 7$ one time each, and $3, 4, 5$ two times each.

Thus, we have $l_1 = 5$, $l_2 = l_3 = 4$, $l_4 = 3$, $l_5 = 2$, and $l_6 = l_7 = 1$. This implies that $\mu = (m_5, m_4, m_3, m_2, m_1) = (7, 5, 4, 3, 1)$.