

Topics in Combinatorics IV, Problems Class 5 (Week 12)

The class was devoted to the counterpart of question 11.1, where P is a regular tetrahedron in $\mathbb{R}^3$ centered at the origin.

5.1. Let P be a regular tetrahedron in $\mathbb{R}^3$ centered at the origin. A *symmetry* of P is $g \in O_3(\mathbb{R})$ taking P to itself.

(a) Show that symmetries of P compose a group, denote it by $\text{Sym } P$.

If $g_1, g_2 \in O_3(\mathbb{R})$ take P to itself, then so do g_i^{-1} and $g_1 g_2$.

(b) Show that $\text{Sym } P$ acts on the set of faces of P transitively.

Denote the vertices of P by $v_1, \dots, v_4$. Then $\text{Sym } P$ contains rotations r_i around the lines passing through v_i and the center of the opposite face. These rotations take any face of P to any other.

(c) Show that $\text{Sym } P$ acts transitively on the set of *flags*, i.e. triples (v, e, f) , where v is a vertex of P , e is an edge, f is a face, and $v \in e \subset f$.

Let (v, e, f) and (v', e', f') be two flags. Using rotations r_i , we can take v to v' . Let $g_1 v = v'$, $g_1 \in \text{Sym } P$, denote $e'' = g_1 e$. If $v' = v_j$, then powers of rotation r_j act transitively on all three edges incident to v_j , so there is $g_2 \in \text{Sym } P$ such that $g_2(v') = v'$ and $g_2(e'') = e'$. So, $g_2 g_1$ takes v to v' and e to e' , denote $f'' = g_2 g_1(f)$. If $f'' = f'$ then we are done. Otherwise, observe that there are precisely two faces containing e' , and they are taken to each other by a reflection in the plane passing through e' and the center of the opposite edge. Applying this reflection, we take f'' to f' , preserving e' and v' .

(d) (this is specific to cube!)

(e) Compute the order of $\text{Sym } P$.

The number of flags is $4 \cdot 3 \cdot 2 = 24$, the group $\text{Sym } P$ acts on the set of flags with a single orbit and trivial stabilizer. Therefore, $|\text{Sym } P| = 24$.

(f) Show that $\text{Sym } P$ is generated by reflections.

We have shown above that to take any flag to any flag we need rotations r_i and reflections. Now, every r_i is a product of two reflections, where the mirrors pass through the axis of r_i and any other vertex of P . There are six such reflections altogether (every mirror passes through an edge and the center of the opposite edge).

(g) Show that $\text{Sym } P$ cannot be generated by two reflections.

If we assume that $\text{Sym } P$ is generated by two reflections r_α and r_β , then the group would leave the space $\{\alpha, \beta\}^\perp$ of positive dimension invariant. However, $\text{Sym } P$ is clearly irreducible.

Let (v, e, f) be a flag. Let $p_1 = v$, denote by p_2 the center of e , by p_3 the center of f , and by O the center of P (i.e., the origin of $\mathbb{R}^3$). Let C be the cone over triangle $p_1p_2p_3$ with apex O , i.e. the intersection of the three halfspaces: we take plane passing through Op_ip_j and take the halfspace containing the third point p_k .

- (h) Show that three reflections in the walls of C generate $\text{Sym } P$. Write down the relations among these generators (i.e., give a presentation of $\text{Sym } P$ by generators and relations, where generators are the three reflections above).

Denote by s_i the reflection in plane Op_jp_k . Then it is easy to see that six reflections mentioned above are $s_1, s_2, s_3, s_1s_2s_1, s_2s_3s_2$, and $s_1s_2s_3s_2s_1$.

We are left to find the relations, i.e. the orders of s_is_j . Reflections s_1 and s_2 generate a dihedral group preserving f , so $(s_1s_2)^3 = \text{id}$. Similarly, reflections s_2 and s_3 generate a dihedral group preserving a neighboring face (which can be obtained from f by applying reflection $s_1s_2s_3s_2s_1$), so $(s_2s_3)^3 = \text{id}$. Finally, the reflections s_1 and s_3 commute (as the planes Op_1p_2 and Op_2p_3 are orthogonal), so

$$\text{Sym } P = \langle s_1, s_2, s_3 \mid s_i^2, (s_1s_2)^3, (s_2s_3)^3, (s_1s_3)^2 \rangle$$

Alternatively, one can use Theorem 7.7 from lectures.