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Topics in Combinatorics IV, Problems Class 7 (Week 16)

7.1. List the roots of root system of type B_2.

The Weyl group is $I_2(4)$, so it contains four reflections. We need to express eight roots as linear combinations of simple roots α_1 and α_2. We assume that $\langle \alpha_1 \mid \alpha_2 \rangle = -2$, i.e. the length of α_1 is bigger that the length of α_2.

As $\langle \alpha_1 \mid \alpha_2 \rangle = -2$ and $\langle \alpha_2 \mid \alpha_1 \rangle = -1$, we see that

$$2 = \frac{\langle \alpha_1 \mid \alpha_2 \rangle}{\langle \alpha_2 \mid \alpha_1 \rangle} = \frac{(\alpha_1, \alpha_1)}{(\alpha_2, \alpha_2)},$$

so we may assume $(\alpha_1, \alpha_1) = 2$ and $(\alpha_2, \alpha_2) = 1$. Then we have

$$(\alpha_1, \alpha_2) = \|\alpha_1\| \|\alpha_2\| (-\cos\frac{\pi}{4}) = \sqrt{2}(-\frac{1}{\sqrt{2}}) = -1.$$

Now, reflecting α_1 with respect to α_2 we get

$$r_{\alpha_2}(\alpha_1) = \alpha_1 - \langle \alpha_1 \mid \alpha_2 \rangle \alpha_2 = \alpha_1 - (-2)\alpha_2 = \alpha_1 + 2\alpha_2,$$

and reflecting α_2 with respect to α_1 we get

$$r_{\alpha_1}(\alpha_2) = \alpha_2 - \langle \alpha_2 \mid \alpha_1 \rangle \alpha_1 = \alpha_2 - (-1)\alpha_1 = \alpha_1 + \alpha_2.$$

Thus, we found four non-collinear roots, so taking negatives we obtain all 8 roots of the root system.

7.2. Let (G, S) be a Coxeter system, and let $l(g)$ denote the length of $g \in G$.

(a) Let $g \in G$, $s \in S$. Show that $|l(gs) - l(g)| = 1$.

(b) Let $g \in G$, $s, t \in S$. Suppose that $l(gs) = l(g) + 1$ and $l(tg) = l(g) + 1$. Show that either $l(tgs) = l(g) + 2$ or $tgs = g$.

The first statement is straightforward. By definition, $l(gs) \le l(g) + 1$, and since $g = (gs)s$, we also have $l(gs) \le l(g) + 1$. Thus $|l(gs) - l(g)| \le 1$. Finally, by Exercise 8.18, $l(gs) \ne l(g)$.

To prove the second statement, choose any reduced expression $w = s_1 \dots s_k$ for g, i.e. $k = l(g)$. By the assumption, both tw and ws are also reduced. If tws is reduced, then $l(tgs) = l(g) + 2$, so we assume that tws is not reduced.

Since ws is reduced but tws is not, we can apply the Exchange Condition, which says that either $ws \sim_G ts_1 \dots \hat{s}_i \dots s_k s$ or $ws = tw$. In the latter case $tws = wss \sim_G w$, so $tgs = g$. In the former case $w \sim_G ts_1 \dots \hat{s}_i \dots s_k$, which implies $tw \sim_G s_1 \dots \hat{s}_i \dots s_k$ in contradiction with tw being reduced.