Durham University
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Epiphany 2026

Topics in Combinatorics IV, Problems Class 8 (Week 18)

Problems class was mostly devoted to HW15-16. The main point is that it is a good idea to use what is known and not to use what is not known.

8.1. Let (G, S) be a Coxeter system, $w = s_{i_1} \dots s_{i_k}$ a reduced word. Let $s_i \in S$, assume that $s_i \in r(w)$. Then $s_i = s_{i_j}$ for some j.

Suppose all letters in w are distinct from s_i. Then $s_i \in r(w)$ is a product of other generators, and thus G is generated by $S \setminus s_i$. Denote $s_j = r_{\alpha_j}$, and define $L = \mathrm{span}_{\mathbb{R}}\{\{\alpha_1, \dots, \alpha_n\} \setminus \alpha_i\}$. Then L has dimension $(n-1)$, and $\alpha_i \notin L$.

Observe that for $j \neq i$ for any $x \in \mathbb{R}^n$ we have $s_j x - x \in L$. Indeed,

$$s_j x - x = (x - \langle x \mid \alpha_j \rangle \alpha_j) - x = -\langle x \mid \alpha_j \rangle \alpha_j \in L.$$

Furthermore, for every $g \in G$ we have $gx - x \in L$. This can be proven by induction on the length of g. The base is proved above, so we need to prove that if $g'x - x \in L$ then $(s_j g')x - x \in L$. Indeed, denote $g'x - x = l \in L$. Then

$$(s_j g')x - x = s_j(g'x) - x = s_j(x + l) - x = (s_j x - x) + s_j l = (s_j x - x) + (s_j l - l) + l = l_1 + l_2 + l \in L,$$

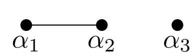
where $l_1, l_2 \in L$. Taking $g = s_i$ and $x = \alpha_i$, we obtain $s_i\alpha_i - \alpha_i \in L$. However, $s_i\alpha_i - \alpha_i = -\alpha_i - \alpha_i = -2\alpha_i \notin L$, so we got a contradiction.

Alternatively, one can observe that by Lemma 8.16 there exists reduced $w_1 \sim_G w$ starting with s_i, and by Matsumoto's Theorem w_1 can be obtained from w by M-reduction. However, elementary M-reductions do not introduce new letters, so if w has no s_i in it then the same should hold for w_1, which leads to a contradiction.

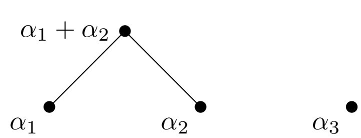
8.2. Draw Hasse diagrams of root posets of root systems of types A_3, $A_2 \oplus A_1$, B_3.

Using explicit form of roots of the root systems, one obtains the following Hasse diagrams.

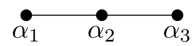
- $A_2 \oplus A_1$
  Disconnected Dynkin diagram looks as follows:



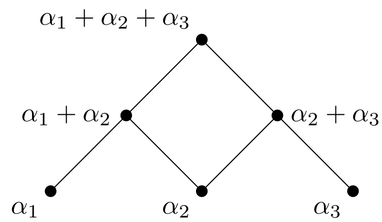
There are four positive roots of $A_2 \oplus A_1$: simple roots $\alpha_1, \alpha_2, \alpha_3$, and also the only non-simple positive root $\alpha_1 + \alpha_2$ of A_2. The Hasse diagram of the root poset is



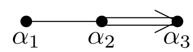
- A_3
  Dynkin diagram looks as follows:



  There are six positive roots of A_3: simple roots $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3 - e_4$, and also $\alpha_1 + \alpha_2 = e_1 - e_3$, $\alpha_2 + \alpha_3 = e_2 - e_4$, and $\alpha_1 + \alpha_2 + \alpha_3 = e_1 - e_4$. Thus, the Hasse diagram of the root poset is



- B_3
  Dynkin diagram looks as follows:



  There are nine positive roots of B_3: simple roots $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \alpha_3 = e_3$, and also $\alpha_1 + \alpha_2 = e_1 - e_3$, $\alpha_2 + \alpha_3 = e_2$, $\alpha_1 + \alpha_2 + \alpha_3 = e_1$, $\alpha_2 + 2\alpha_3 = e_2 + e_3$, $\alpha_1 + \alpha_2 + 2\alpha_3 = e_1 + e_3$, $\alpha_1 + 2\alpha_2 + 2\alpha_3 = e_1 + e_2$. Thus, the Hasse diagram of the root poset is