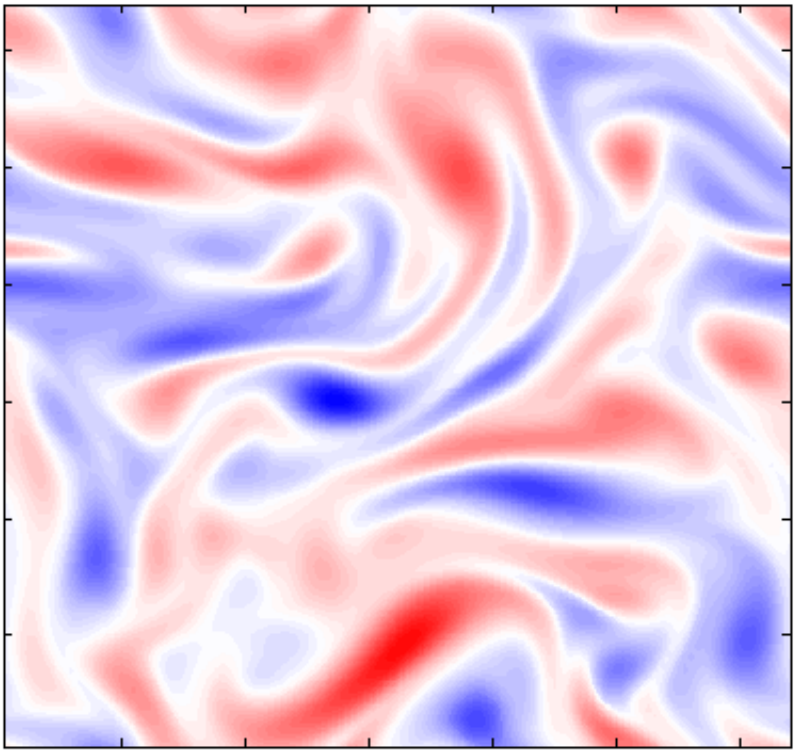
Fluid Mechanics III/IV

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Contents

0 Overview **3**

1 Fluid kinematics **4**

1.1 Lagrangian and Eulerian pictures 4
1.2 Streamlines and stream functions 7
1.3 Streaklines 9
1.4 The Jacobian Matrix 10
1.5 Compressibility 10
1.6 Local analysis of the flow 12
1.7 Vorticity and circulation 14
1.8 Flow reconstruction 16
1.9 Integrals over material domains 18

2 Dynamics of ideal fluids **19**

2.1 Conservation of mass 20
2.2 Conservation of momentum for an ideal fluid 21
2.3 Incompressible Euler equations 23
2.4 Energy 25
2.5 Vorticity dynamics 27
2.6 2D vorticity evolution 28
2.7 3D vorticity evolution 29
2.8 Helicity 30
2.9 Circulation 32

3 Water waves **33**

3.1 Free surface boundary conditions 33
3.2 Governing equations 34
3.3 Linearisation 34
3.4 Travelling wave solutions 35
3.5 General solution 37
3.6 Standing waves 38

0 Overview

Continuum mechanics models the behaviour of materials (solids/liquids/gases) by modelling them as a continuous mass rather than discrete particles.

In this course we will focus on *fluids*, meaning liquids and gases. The subject of *elasticity* deals with the behaviour of solids, although much of the mathematics is similar. The theory we will develop will deal with observable properties such as velocity, density and pressure of the fluid. These quantities are understood to represent averages over small volumes within the flow which are large compared to the molecules they contain, but "infinitesimal" with respect to the properties of the overall fluid flow. We shall use the term *fluid parcel* to indicate a small volume within the flow, and *fluid particle* to indicate a point of the fluid continuum.

Fluid flow may be classified in various ways:

1. *Viscous* vs. *inviscid*. Broadly speaking, viscosity (internal friction between fluid particles) is important near solid surfaces, but can be ignored elsewhere.
2. *Turbulent* vs. *laminar*. Typically high speed or (nearly) inviscid flows are turbulent/chaotic, while slower or more viscous flows are laminar (organised).
3. *Rotational* vs. *irrotational*. Rotational fluid flows have *vorticity* (angular momentum within the fluid), whereas irrotational flows do not. Typically, vorticity is generated on boundaries, and propagates into the interior of the fluid.

Historically, fluid mechanics tended to focus on irrotational flows, since these are mathematically simpler. Unfortunately, most real fluids are rotational, and often turbulent too. Vorticity is usually the best way to understand such flows.

▶ We will deal purely with classical mechanics, since the majority of fluids of practical importance are neither quantum nor relativistic. The mathematical and physical problems are quite challenging enough even in purely classical fluids!

The course will be organised as follows:

Michaelmas term:

1. **Fluid kinematics** – how to describe the motion of fluids.
2. **Dynamics of ideal fluids** – equations of motion for inviscid fluids (think liquids) and their properties.
3. **Water waves** – an important example of how to solve the (inviscid) equations of motion by making simplifying assumptions.

Epiphany term:

1. **Compressible flow** – modifying the ideal equations of motion to model gases, and the origin of sound waves.
2. **Instability** – this section aims to explain how laminar flows break down and turbulence is generated (modelling turbulence itself is beyond our scope!).
3. **Dynamics of viscous fluids** – equations of motion for viscous fluids, and the dynamical consequences of viscosity.

The pre-requisites are a passing knowledge of Newtonian mechanics (Dynamics I) and a good knowledge of vector calculus (Analysis in Many Variables II). Before starting the course it is *strongly advised that you go over the material in the revision document*. This document alongside the formula sheet should be used for reference throughout the course.

1 Fluid kinematics

In this section we will introduce the mathematical tools and concepts needed to describe fluid motion. In particular, we need some more advanced vector calculus than was covered in AMV. The dynamical equations governing the motion will come later.

In our *continuum* model a fluid is a continuous medium made up of *fluid parcels*. Each infinitesimal fluid parcel contains an average of the properties of the fluid inside it and interacts with its neighbours via surface contact (more on this later). *Fluid particles* are points within the continuum that are moved by the fluid parcel motion. Essentially, they are along for the ride.

Example → Fluid particle simulation.
https://experiments.withgoogle.com/fluid-particles

In general the continuous media comprising our fluid will occupy some open set $D_t \subset \mathbb{R}^N$, with $N = 2$ or 3. As the fluid moves, the shape of D_t will evolve.

1.1 Lagrangian and Eulerian pictures

If we label a fluid particle by its initial position $\boldsymbol{a} \in D_0$, then its position at any later time is described by its *trajectory* or *particle path*, $\boldsymbol{x}(\boldsymbol{a}, t)$. This function is assumed to be continuous and invertible, so that it may be inverted to give the initial coordinate $\boldsymbol{a}(\boldsymbol{x}, t)$ of the particle located at $\boldsymbol{x}$.

▶ For formulating differential equations, we will assume that the mapping $\boldsymbol{a} \to \boldsymbol{x}(\boldsymbol{a}, t)$ is at least twice continuously differentiable for any time t.

Hence at any time t we may use either the coordinates $\boldsymbol{x}$ or $\boldsymbol{a}$ to describe the fluid:

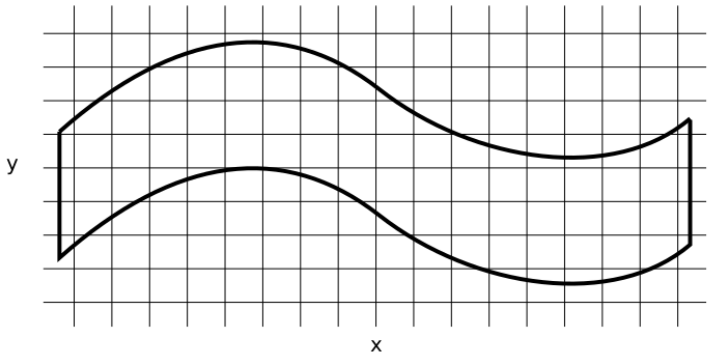
1. *Eulerian picture* - fluid particles are labelled by their current position $\boldsymbol{x}$.
2. *Lagrangian picture* - fluid particles are labelled by their original position $\boldsymbol{a}$.

As time evolves, the Lagrangian picture follows the particle that was situated at $\boldsymbol{a}$ at $t = 0$. Whereas the Eulerian picture samples fluid particles at fixed $\boldsymbol{x}$, thereby measuring quantities on different particles but at a fixed point in space. We define time derivatives appropriate to each of these pictures:

1. Eulerian: $\partial/\partial t := (\partial/\partial t)_{\boldsymbol{x}}$ is the derivative with respect to time keeping $\boldsymbol{x}$ constant.
2. Lagrangian: $\mathrm{D}/\mathrm{D}t := (\partial/\partial t)_{\boldsymbol{a}}$ is the derivative with respect to time keeping $\boldsymbol{a}$ constant.

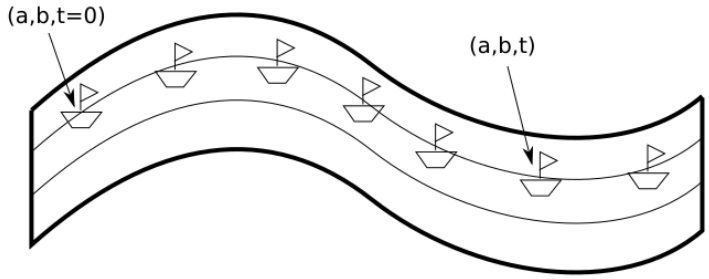
So, for any function f, $\partial f/\partial t$ is the rate of change of f at a fixed position while $\mathrm{D}f/\mathrm{D}t$ is the rate of change of f when moving with the fluid. This new derivative is called the *material* or *convective* derivative. If f remains constant as it moves with the fluid, i.e. $\mathrm{D}f/\mathrm{D}t = 0$, then it is known as a *material scalar* or *Lagrange invariant*.

Example → Consider measuring the temperature of water in a river. The Eulerian picture is equivalent to setting up a grid and measuring the temperature at each point in the grid throughout time $T(x, y, t)$. e.g.



At each time we sample a different fluid element. $\left(\frac{\partial T(x,y,t)}{\partial t}\right)_{\boldsymbol{x}}$ gives the rate of change of temperature measured at a given position.

The Lagrange picture is equivalent to dropping in many boats at specific locations (a, b) which we recover later. The boats sit on the same fluid element as they ride along and record how its temperature changes with time. e.g.



$\frac{\mathrm{D}T}{\mathrm{D}t} = \left(\frac{\partial T(a,b,t)}{\partial t}\right)_{\boldsymbol{a}}$ gives the change in temperature of the fluid element as it flows along after starting at a given location (a, b).

Example → Vortex merging simulation.
`https://www.youtube.com/watch?v=debN4toiYBM`

The *velocity* of a fluid particle is defined as the material derivative of its position:

$$\boldsymbol{u}\big(\boldsymbol{x}(\boldsymbol{a},t),t\big) := \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t} = \frac{\partial}{\partial t}\boldsymbol{x}(a,b,c,t). \tag{1.1}$$

Looking at it from the Eulerian viewpoint, $\boldsymbol{u}(\boldsymbol{x}, t)$ at time t is a vector field that tells us the instantaneous velocity of the fluid at every position $\boldsymbol{x}$. Given the Eulerian velocity field, we can find the Lagrangian coordinates by solving the initial value problem defined by the above system of ODEs.

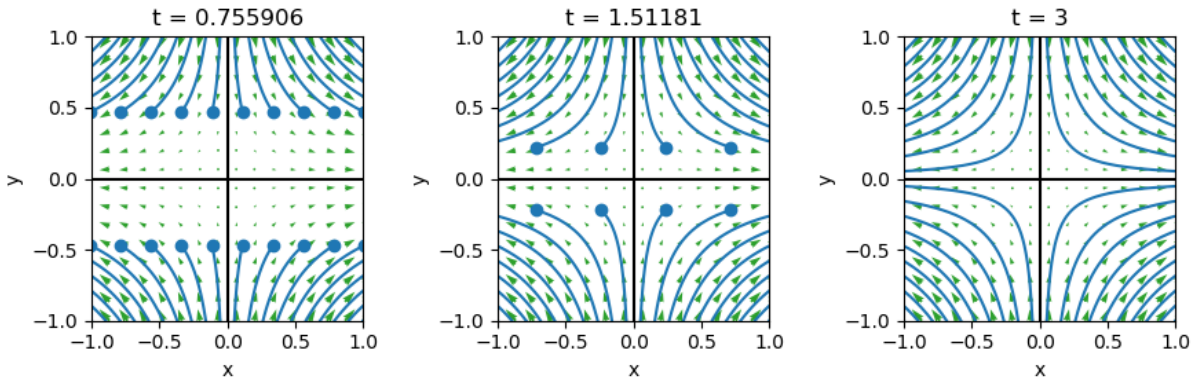
▶ We will see that $\boldsymbol{u}(\boldsymbol{x}, t)$, and not $\boldsymbol{x}(\boldsymbol{a}, t)$, is treated as the primary variable in fluid dynamics.

Example → Let $D_t = \mathbb{R}^2$. Find the particle paths in the velocity field $\boldsymbol{u}(x, y, t) = (x, -y)$.
The path $\boldsymbol{x}(\boldsymbol{a}, t) = (x, y)$ of a given particle is found by fixing $\boldsymbol{a} = (a, b)$ so that (1.1) becomes a set of ODEs in the variable t, just like in particle mechanics:

$$\frac{dx}{dt} = x, \quad x(0) = a, \qquad \frac{dy}{dt} = -y, \quad y(0) = b. \tag{1.2}$$

Integrating gives

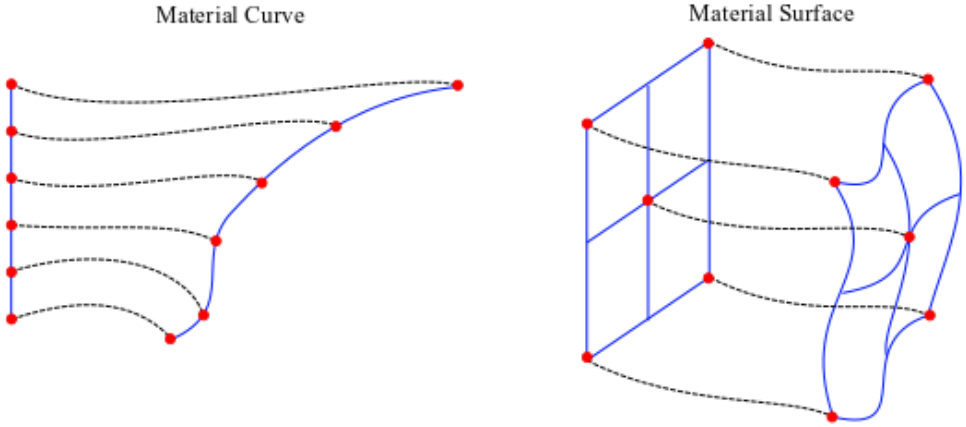
$$\boldsymbol{x}(\boldsymbol{a}, t) = \big(a\mathrm{e}^{t}, b\mathrm{e}^{-t}\big). \tag{1.3}$$



Notice that $xy = ab$, so the particle paths are hyperbolas. Notice also that $\boldsymbol{u} = \boldsymbol{0}$ at $\boldsymbol{x} = \boldsymbol{0}$, which is called a *stagnation point*.

▶ Notice what happens for large time in the previous example: a finite amount of initial fluid will be squashed onto the line $y = 0$ by this flow (a "stable manifold" for those taking Dynamical Systems). Often the Lagrangian viewpoint is used to study underlying "structures" like this in fluid flows. This requires numerics and is beyond the scope of this course.

The fluid elements above remain in a straight line as they move inwards. We could describe this behaviour with a continuous line (rather than individual particles) that moves with the fluid. In general, a curve that threads a set of fluid elements and moves with them in time is known as a *material curve* or *material line*. By analogy a surface that moves with the flow is known as a *material surface*.



▶ Even a simple $\boldsymbol{u}$ can have *chaotic* particle paths that depend very sensitively on $\boldsymbol{a}$. One example is the *blinking vortex* flow generated by two overlapping vortices that are switched on and off alternately.

A very important formula relating the Eulerian and Lagrangian time derivatives is derived by applying the chain rule:

$$\begin{aligned}\frac{\mathrm{D}f}{\mathrm{D}t} &= \frac{\partial}{\partial t} f\big(\boldsymbol{x}(\boldsymbol{a},t),t\big) \\ &= \left(\frac{\partial f}{\partial t}\right)_{\boldsymbol{x}} + \frac{\partial f}{\partial x_i}\left(\frac{\partial x_i}{\partial t}\right)_{\boldsymbol{a}} \\ &= \frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} \\ &= \frac{\partial f}{\partial t} + \boldsymbol{u}\cdot\nabla f. \end{aligned} \tag{1.4}$$

This can then be generalised to vector quantities by applying it to each vector element in turn:

$$\frac{\mathrm{D}\mathbf{f}}{\mathrm{D}t} = \frac{\partial \mathbf{f}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\mathbf{f}. \tag{1.5}$$

Note that the l.h.s is in *Lagrangian* coordinates, while the r.h.s is in *Eulerian* coordinates. However, provided we know $\boldsymbol{x}(\boldsymbol{a})$ we can make each comparable.

Example → In the flow $\boldsymbol{u} = (x, -y)$, let $f(x, y, t) = y^2 t$. Then we can check

$$\frac{\partial f}{\partial t} = y^2, \qquad \boldsymbol{u}\cdot\nabla f = -y\frac{\partial}{\partial y}(y^2 t) = -2y^2 t, \qquad \frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial}{\partial t}(b^2 t\mathrm{e}^{-2t}) = b^2\mathrm{e}^{-2t} - 2b^2 t\mathrm{e}^{-2t} = y^2 - 2y^2 t.$$

1.2 Streamlines and stream functions

A *streamline* is an integral curve of the Eulerian velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ at a fixed time. It is not the same as a particle path, but is often used as a way to visualise a snapshot of the velocity field. A streamline is a curve $\boldsymbol{x}(s)$ satisfying the ODEs

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{u}\big(\boldsymbol{x}(s), t\big), \tag{1.6}$$

for fixed t, i.e. *t is a constant* when integrating to find the streamlines.

Example → Find the streamlines of the velocity field $\boldsymbol{u} = (1, t)$ for $D_t = \mathbb{R}^2$.
We have to solve two simultaneous ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}s} = 1, \quad x(0) = x_0, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = t, \quad y(0) = y_0.$$

We use x_0 and y_0 so as not to confuse these with particle paths. Integrating the first equation shows that $x = s + x_0$, while the second gives $y = ts + y_0$. Eliminating s shows that the streamlines are straight lines with slope t,

$$y = t(x - x_0) + y_0.$$

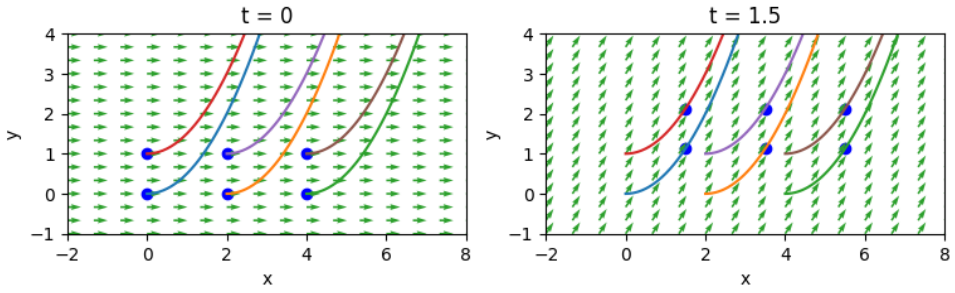
Notice that the pattern of streamlines is changing over time, because this flow is *unsteady* (i.e. $\boldsymbol{u}$ depends on t). To find the particle paths, we need to solve

$$\frac{dx}{dt} = 1, \quad x(0) = a, \qquad \frac{dy}{dt} = t, \quad y(0) = b.$$

We have $x = t + a$, but $y = \frac{1}{2}t^2 + b$. Eliminating t we see that the particle paths are parabolas,

$$y = \tfrac{1}{2}(x - a)^2 + b.$$

Below are the particle paths for 6 different starting positions alongside arrows showing the flow direction at each time. The streamlines are instantaneous tangents to any given particle path.



▶ In a steady flow, the particle paths and streamlines would coincide.

If $\boldsymbol{u}$ is both two-dimensional and divergence-free ($\nabla \cdot \boldsymbol{u} = 0$), then we can write

$$\boldsymbol{u}(x, y, t) = \nabla \times \Big(\psi(x, y, t)\boldsymbol{e}_z\Big), \tag{1.7}$$

for some scalar function $\psi(x, y, t)$ called a *stream function*. The benefit of this is that (a) we only have to deal with a scalar function instead of a vector, and (b) it is easy to plot the streamlines, because of the following.

Proposition 1.1. *The streamlines of $\boldsymbol{u} = \nabla \times (\psi \boldsymbol{e}_z)$ are contours of ψ.*

Proof. From AMV we know that $\boldsymbol{u} = \nabla\psi \times \boldsymbol{e}_z$, which implies that $\boldsymbol{u} \cdot \nabla\psi = 0$. But the gradient $\nabla\psi$ is perpendicular to the contours of ψ, so $\boldsymbol{u}$ must be tangent to these contours. Therefore they are streamlines. □

Example → Find a stream function for $\boldsymbol{u} = (x, -y)$.
This velocity field is divergence free, and can therefore be written in terms of a stream function. To find the stream function we need to simultaneously solve

$$\frac{\partial \psi}{\partial y} = x, \quad \frac{\partial \psi}{\partial x} = y.$$

Integrating the first equation with respect to y gives $\psi = xy + g(x)$ for some arbitrary function $g(x)$. Differentiating with respect to x then gives

$$\frac{\partial \psi}{\partial x} = y + g'(x)$$

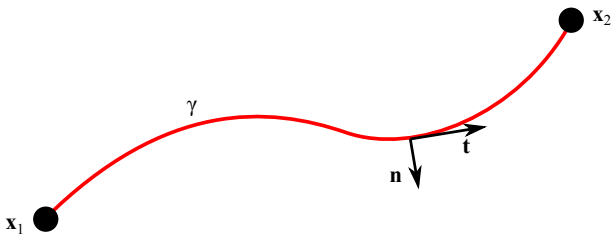
and comparing to the required equation shows that $g'(x) = 0$. Thus $g(x) = c$ (constant) and hence

$$\psi(x, y) = xy + c.$$

The contours of ψ are therefore parabolas as expected. The arbitrary constant c has no effect on $\boldsymbol{u}$ which is the curl of ψ.

Proposition 1.2. *The volume flow rate between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ (per unit height) is $\psi(\boldsymbol{x}_2) - \psi(\boldsymbol{x}_1)$.*

Proof. Let γ be any simple curve connecting $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$, with tangent vector $\boldsymbol{t}(l)$.



The normal to γ may be written as $\boldsymbol{n} = \dfrac{(t_y, -t_x)}{|\boldsymbol{t}|}$, so the flow rate per unit height through γ is

$$\int_\gamma \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}l = \int_\gamma \frac{(t_y, -t_x) \cdot (\partial\psi/\partial y, -\partial\psi/\partial x)}{|\boldsymbol{t}|}\, \mathrm{d}l \tag{1.8}$$

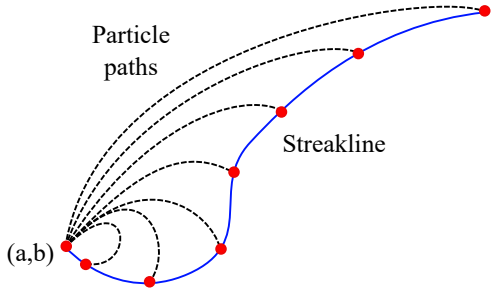
$$= \int_\gamma \frac{\boldsymbol{t} \cdot \nabla\psi}{|\boldsymbol{t}|}\, \mathrm{d}l = \int_\gamma \nabla\psi \cdot \mathrm{d}\boldsymbol{l} = \psi(\boldsymbol{x}_2) - \psi(\boldsymbol{x}_1). \tag{1.9}$$

□

▶ Notice that this is independent of the choice of γ, i.e. it measures the *net* flow rate.

1.3 Streaklines

A *streakline* is curve made up of all fluid elements that pass through a given point. In practise these are often what are used to visualise flow in experiments, e.g. smoke trails in wind tunnels or ink injected into water. Streaklines are comprised of a series of fluid elements, each on its own particle path, that passed through a given point (a,b).



To find a streakline we need to find the general expression for particle paths in the flow and then parameterise them so as to pick out the ones that passed through (a,b).

Example → Find the streakline of $\boldsymbol{u} = (1, t)$ for $D_t = \mathbb{R}^2$ that originates from (a,b) at $t = 0$. The particle paths in general are given by

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = t.$$

Integrating gives

$$x(t) = t + A, \qquad y(t) = \frac{t^2}{2} + B,$$

where A and B are constants we will now set. We now introduce $\xi \in [0, t]$ such that one of the chain of fluid elements that forms the streakline passes through (a,b) at $t = \xi$. Giving

$$a = \xi + A \qquad b = \frac{\xi^2}{2} + B$$

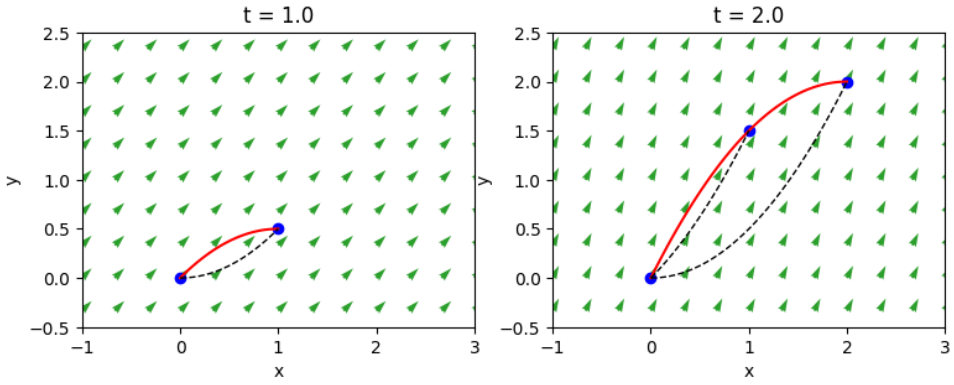
Rearranging gives $A = a - \xi$ and $B = b - \xi^2/2$. Substituting this into the particle paths gives the line of fluid elements which passed through (a,b) between 0 and t.

$$x = t - \xi + a, \qquad y = \frac{t^2}{2} - \frac{\xi^2}{2} + b$$

Note the parametric variable that traces out the curves described by x and y is now ξ. t is taken to be constant and sets the length of time the streakline has formed for. Sometimes we can eliminate ξ to give the explicit form for the equation of the streakline. Using the first equation here gives

$$y = \frac{t^2}{2} + b - \frac{1}{2}(t + a - x)^2.$$

where $x \in [a, t + a]$ (since $\xi \in [0, t]$). Below is the streakline (red) formed by particles injected at $(0, 0)$ from $t = 0$ onwards. The blue dots show a few individual particles and the dashed lines show their respective paths.



▶ For steady flows streamlines, particle paths and streaklines all coincide.

1.4 The Jacobian Matrix

The particle paths that define our Lagrange coordinate system represent a one-to-one mapping (or function) of the initial coordinate $\boldsymbol{a} = (a, b, c)$ to its later position $\boldsymbol{x} = (x, y, z)$. Accordingly, we may define the Jacobian matrix of this mapping

$$\mathbf{J}_{ij} = \frac{\partial x_i}{\partial a_j}, \tag{1.10}$$

which when applied to a line segment connecting two adjacent particles at $t = 0$, defines how this segment evolves to later times via $dl_i = \mathbf{J}_{ij} da_j$. Importantly for what comes next, the determinant of this matrix (J, known as the *Jacobian*) given by

$$J = \frac{\partial(x, y, z)}{\partial(a, b, c)} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}, \tag{1.11}$$

is the factor by which the volume of an infinitesimal fluid parcel varies in time:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = J\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c. \tag{1.12}$$

Our assumption that the mapping is invertible implies that $J \neq 0$.

1.5 Compressibility

A flow is *incompressible* if it preserves the volume of every material subregion D_t, i.e.

$$\int_{D_t} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{D_0} \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c. \tag{1.13}$$

Changing variables from $\boldsymbol{x}$ to $\boldsymbol{a}$ on the left hand side, we have

$$\int_{D_0} J\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \int_{D_0} \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c, \tag{1.14}$$

so a flow is incompressible if and only if $J \equiv 1$.

Next, we show that compressibility is related to the divergence of $\boldsymbol{u}$. This relies on the following.

Lemma 1.3. *The relative change in J moving with the fluid is*

$$\frac{1}{J}\frac{\mathrm{D}J}{\mathrm{D}t} = \nabla \cdot \boldsymbol{u}. \tag{1.15}$$

Note that the l.h.s. of this equation is in *Lagrangian coordinates*, while the r.h.s is in *Eulerian*. However, provided we have our particle paths $\boldsymbol{x}(\boldsymbol{a})$ each side can be made comparable.

Proof. This is a little cumbersome, so let us outline the steps for $N = 2$, leaving $N = 3$ as an exercise. By definition,

$$J = \frac{\partial x_1}{\partial a_1}\frac{\partial x_2}{\partial a_2} - \frac{\partial x_2}{\partial a_1}\frac{\partial x_1}{\partial a_2}, \tag{1.16}$$

so

$$\frac{\mathrm{D}J}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x_1}{\partial a_1}\right)\frac{\partial x_2}{\partial a_2} + \frac{\partial x_1}{\partial a_1}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x_2}{\partial a_2}\right) - \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x_2}{\partial a_1}\right)\frac{\partial x_1}{\partial a_2} - \frac{\partial x_2}{\partial a_1}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x_1}{\partial a_2}\right). \tag{1.17}$$

Since $\dfrac{\mathrm{D}}{\mathrm{D}t} = \left(\dfrac{\partial}{\partial t}\right)_{\boldsymbol{a}}$ treats $\boldsymbol{a}$ as constant, we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x_i}{\partial a_j}\right) = \frac{\partial}{\partial a_j}\left(\frac{\mathrm{D}x_i}{\mathrm{D}t}\right) = \frac{\partial u_i}{\partial a_j} = \frac{\partial u_i}{\partial x_1}\frac{\partial x_1}{\partial a_j} + \frac{\partial u_i}{\partial x_2}\frac{\partial x_2}{\partial a_j}. \tag{1.18}$$

Substituting into (1.17) and cancelling like terms (left as an excerise) gives

$$\frac{\mathrm{D}J}{\mathrm{D}t} = \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\left(\frac{\partial x_1}{\partial a_1}\frac{\partial x_2}{\partial a_2} - \frac{\partial x_2}{\partial a_1}\frac{\partial x_1}{\partial a_2}\right) = \Big(\nabla \cdot \boldsymbol{u}\Big)J. \tag{1.19}$$

□

Proposition 1.4. *A flow is incompressible ($J \equiv 1$) if and only if $\nabla \cdot \boldsymbol{u} \equiv 0$.*

Proof. It follows immediately from Lemma 1.3 that $J \equiv 1$ implies $\nabla \cdot \boldsymbol{u} \equiv 0$. To show the converse, assume that $\nabla \cdot \boldsymbol{u} \equiv 0$ and integrate (1.15) over D_0 to find

$$0 = \int_{D_0} \frac{\mathrm{D}J}{\mathrm{D}t}\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_0} J\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c = \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{1.20}$$

Hence the volume of D_t is unchanging for any D_0. □

Example $\to \boldsymbol{u}(x, y) = (x, -y)$.
We found the particle paths $\boldsymbol{x}(\boldsymbol{a}, t) = (a\mathrm{e}^{t}, b\mathrm{e}^{-t})$, so the Jacobian is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial a} & \dfrac{\partial x}{\partial b} \\ \dfrac{\partial y}{\partial a} & \dfrac{\partial y}{\partial b} \end{vmatrix} = \begin{vmatrix} \mathrm{e}^{t} & 0 \\ 0 & \mathrm{e}^{-t} \end{vmatrix} = 1,$$

consistent with the fact that $\nabla \cdot \boldsymbol{u} = 0$.

Example $\to \boldsymbol{u}(x, y) = (U_0, -y)$.
For this compressible flow $\nabla \cdot \boldsymbol{u} = -1$. The particle paths are $\boldsymbol{x}(\boldsymbol{a}, t) = (U_0 t + a, b\mathrm{e}^{-t})$, so the Jacobian is

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial a} & \dfrac{\partial x}{\partial b} \\ \dfrac{\partial y}{\partial a} & \dfrac{\partial y}{\partial b} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & e^{-t} \end{vmatrix} = e^{-t},$$

Thus,

$$\frac{1}{J}\frac{\mathrm{D}J}{\mathrm{D}t} = -\frac{e^{-t}}{e^{-t}} = -1,$$

as expected.

▶ Broadly speaking, liquids are incompressible but gases are compressible.

1.6 Local analysis of the flow

To get a physical picture of fluid motion, it is useful to compare the velocity field at two nearby points $\boldsymbol{x}$ and $\boldsymbol{x}+\delta\boldsymbol{x}$. Expanding about the (fixed) point $\boldsymbol{x}$ using Taylor's theorem gives

$$\boldsymbol{u}(\boldsymbol{x}+\delta\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t) + \big[\nabla\boldsymbol{u}(\boldsymbol{x})\big]\cdot\delta\boldsymbol{x} + \mathrm{O}(|\delta\boldsymbol{x}|^2). \tag{1.21}$$

So the local structure of the velocity field is determined by the *velocity gradient tensor* $\nabla\boldsymbol{u}$. This is a matrix or *tensor* with components

$$(\nabla\boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j}, \tag{1.22}$$

Like any rank-two tensor, it may be decomposed as

$$\nabla\boldsymbol{u} = \underbrace{\tfrac{1}{2}\Big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top\Big)}_{\boldsymbol{e}} + \underbrace{\tfrac{1}{2}\Big(\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^\top\Big)}_{\boldsymbol{\Omega}}, \tag{1.23}$$

where $\boldsymbol{e}$ is a symmetric tensor (called the *rate-of-strain tensor*) and $\boldsymbol{\Omega}$ is an anti-symmetric tensor.

Example → Find $\boldsymbol{e}$ and $\boldsymbol{\Omega}$ for the planar shear flow $\boldsymbol{u} = (y, 0)$.

$$\boldsymbol{e} = \frac{1}{2}\begin{pmatrix} \frac{\partial u_x}{\partial x}+\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y}+\frac{\partial u_y}{\partial y}\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},$$
$$\boldsymbol{\Omega} = \frac{1}{2}\begin{pmatrix} \frac{\partial u_x}{\partial x}-\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x} \\ \frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y}-\frac{\partial u_y}{\partial y}\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.$$

To understand what these components mean physically, fix $\boldsymbol{x}$ and consider the Lagrangian motion of a small surrounding fluid element D_t. To linear order, the position of $\boldsymbol{x}+\delta\boldsymbol{x} \in D_t$ evolves according to

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{x}+\delta\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}+\delta\boldsymbol{x},t) \approx \boldsymbol{u}(\boldsymbol{x},t) + \big[\boldsymbol{e}(\boldsymbol{x}) + \boldsymbol{\Omega}(\boldsymbol{x})\big]\cdot\delta\boldsymbol{x}, \tag{1.24}$$

where we have used (1.21). So there are three components:

1. Bodily translation of the whole of D_t with velocity $\boldsymbol{u}(\boldsymbol{x},t)$.

2. Stretching/compression of D_t. Since $\boldsymbol{e}$ is a symmetric matrix, we can find an orthonormal basis $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ in which $\boldsymbol{e}$ is diagonal. These are called the *principal axes*, and the eigenvalues are the corresponding *principal rates of strain*.

3. Rotation of D_t. To see this, note that the matrix $\boldsymbol{\Omega}$ in three dimensions takes the form

$$\boldsymbol{\Omega} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0\end{pmatrix}, \tag{1.25}$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ (an important quantity known as the *vorticity* which we will come back to in the next section). Moreover, we can write

$$\boldsymbol{\Omega} \cdot \delta\boldsymbol{x} = \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x}) \times \delta\boldsymbol{x}, \tag{1.26}$$

showing that the fluid element is rotating with angular velocity $\frac{1}{2}\boldsymbol{\omega}$.

Example → Considering the flow about a local point $\boldsymbol{x} + \delta\boldsymbol{x} = (x_0, y_0) + (\delta x, \delta y)$ for the above example we find

$$\begin{aligned}
\boldsymbol{u}(\boldsymbol{x}, t) &= y_0 \boldsymbol{e}_x \\
\boldsymbol{e}(\boldsymbol{x}) \cdot \delta\boldsymbol{x} &= \frac{1}{2}(\delta y \boldsymbol{e}_x + \delta x \boldsymbol{e}_y) \\
\boldsymbol{\Omega}(\boldsymbol{x}) \cdot \delta\boldsymbol{x} &= \frac{1}{2}(\delta y \boldsymbol{e}_x - \delta x \boldsymbol{e}_y)
\end{aligned}$$

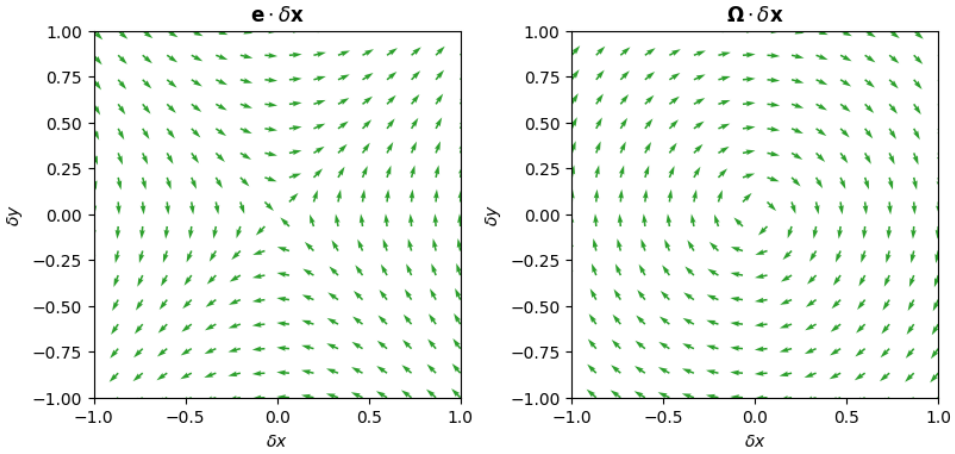
The flow element at (x_0, y_0) is bodily advected at a rate y_0 in the x direction and is both stretched and rotated by the fields shown in the figure below. The eigenvalues of $\boldsymbol{e}$ describing the stretching are $\lambda = \pm\frac{1}{2}$, and the eigenvectors (the principle axes) are

$$\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So locally the flow will stretch a fluid element in the $\boldsymbol{e}_1$ direction, and compress it in the $\boldsymbol{e}_2$ direction (recall the particle trajectories section). An initially circular fluid element will be deformed into an ellipse.

► Since this flow is incompressible, we must have $\mathrm{Tr}(\boldsymbol{e}) = \lambda_1 + \lambda_2 = 0$, and the volume of D_t must be preserved. For compressible flows this does not have to be the case.

The angular velocity for the rotation is $\frac{1}{2}\boldsymbol{\omega} = -\frac{1}{2}\boldsymbol{e}_z$ from which we find $\frac{1}{2}\boldsymbol{\omega} \times \delta\boldsymbol{x} = \frac{1}{2}(\delta y \boldsymbol{e}_x - \delta x \boldsymbol{e}_y)$ as expected.



► The decomposition (1.24) is the fluid version of rigid body dynamics, where the motion of a body is described by translation and rotation. In fluids, we need a third term for stretching/compression.

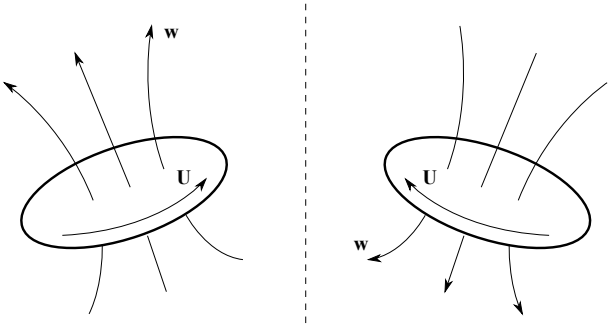
1.7 Vorticity and circulation

The pseudo-vector

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{1.27}$$

is called the *vorticity* and plays a fundamental role in fluid mechanics as we will see later on.

▶ The name *pseudo-vector/axial vector* means that the components of $\boldsymbol{\omega}$ transform like a normal vector under a rotation, but flip sign under a reflection.

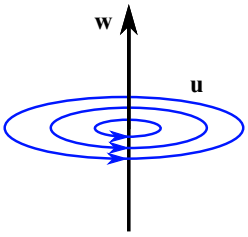


Example → In the two-dimensional example above, $\boldsymbol{\omega} = (0, 0, -1)$ with only one component. As we showed, since $\boldsymbol{\omega} \neq \mathbf{0}$, fluid elements in the shear flow are rotating even though the velocity of each fluid particle is purely in a straight line(!)

Example → The *line vortex* with $\boldsymbol{u}(r,t) = \dfrac{k}{r}\boldsymbol{e}_\theta$ in cylindrical coordinates (r, θ, z).
Away from $r = 0$ we have

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ u_r & ru_\theta & u_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \partial_r & \partial_\theta & \partial_z \\ 0 & k & 0 \end{vmatrix} = 0.$$

So a line vortex has no vorticity – and hence no local rotation – except at $r = 0$ where $\boldsymbol{\omega}$ is infinite.



▶ A more realistic model is the *Rankine vortex* which has a finite inner region where the vorticity is contained [see problem sheet]. A Rankine vortex is sometimes used as a simple model for tornadoes and whirlpools.

Let γ be a simple, smooth, closed curve in the fluid. Then the *circulation* around γ is

$$C_\gamma = \oint_\gamma \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.28}$$

If S_γ denotes a surface spanning γ_t (with orientation chosen from that of C_γ using the right-hand rule) then Stokes' theorem gives

$$C_\gamma = \int_S \nabla \times \boldsymbol{u} \cdot \, \mathrm{d}\boldsymbol{S} = \int_S \boldsymbol{\omega} \cdot \, \mathrm{d}\boldsymbol{S}. \tag{1.29}$$

So the circulation is a measure of the overall rotation of the fluid in a certain region.

Example → Line vortex again.
Let γ be the circle $r = R$, then

$$C_\gamma = \int_0^{2\pi} \frac{k}{R} R \, \mathrm{d}\theta = 2\pi k.$$

This is independent of R and called the "strength" of the line vortex. Notice how there is a non-zero circulation around γ even though $\boldsymbol{\omega} = \mathbf{0}$ on γ itself.

We can use this to see that the singularity in $\boldsymbol{\omega}$ has the form of a Dirac δ-function. Recall that the two-dimensional δ-function $\delta(x, y)$ is a generalised function that vanishes everywhere except at $x = y = 0$ and satisfies

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) \, \mathrm{d}x\mathrm{d}y = 1 \quad \Longleftrightarrow \quad \int_0^{\infty}\int_0^{2\pi} \delta(r, \theta) \, r\mathrm{d}r\mathrm{d}\theta = 1.$$

For the line vortex, we have

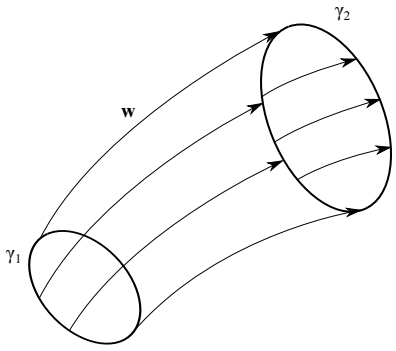
$$\int_0^{\infty}\int_0^{2\pi} \omega_z(r, \theta) \, r\mathrm{d}r\mathrm{d}\theta = \lim_{R\to\infty} C_\gamma = \lim_{R\to\infty} 2\pi k = 2\pi k,$$

constant. So we can write $\boldsymbol{\omega} = 2\pi k \delta(r)\boldsymbol{e}_z$.

Just as a streamline is an integral curve of $\boldsymbol{u}(\boldsymbol{x}, t)$ at a fixed time, a *vortex line* is an integral curve of $\boldsymbol{\omega}(\boldsymbol{x}, t)$ at a fixed time that allows us to visualise the vorticity of the flow. However, it is actually often more useful to consider a *vortex tube*, which is a bundle of vortex lines passing through some reducible closed curve γ.

An important property of a vortex tube is that it has a well-defined "strength", given by its circulation C_γ. This strength should be invariant along its length. The constant strength of the line vortex above is one example of this invariance, but more generally:

Proposition 1.5 (Helmholtz)**.** *If γ_1 and γ_2 are two closed curves encircling the same vortex tube, then $C_{\gamma_1} = C_{\gamma_2}$.*



Proof. This is a straightforward consequence of Stokes' theorem and the divergence theorem. Let S be the tubular surface bounded by γ_1 and γ_2 and composed of vortex lines, so that $\nabla \times \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S} = 0$ on the side wall of the tube. Then the divergence theorem gives that

$$\int_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{\omega} \, \mathrm{d}V = 0.$$

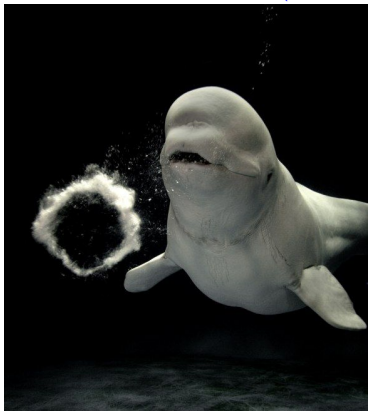
Stokes' theorem applied to the surface and end caps of the tube then shows that

$$\int_S \nabla \times \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S} = \oint_{\gamma_1} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} - \oint_{\gamma_2} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = 0, \tag{1.30}$$

or equivalently $C_{\gamma_1} - C_{\gamma_2} = 0$. The minus sign arises from $\mathrm{d}\boldsymbol{S}$ pointing outwards from the volume on both end surfaces. □

▶ This was proven in a famous 1858 paper by Hermann von Helmholtz introducing "vortex filaments".

Example → Vortex rings.
The above implies that vortex tubes either close back on themselves or stretch out to infinity. It also implies that the vorticity can be increased within a tube by narrowing its width. This is known as vortex stretching (more on this later). A classic example of vortex lines closing in a loop are vortex rings (smoke rings, bubble rings).




1.8 Flow reconstruction

As vortex rings demonstrate it can often be easier to think about fluid flow in terms of the vorticity structures present rather than the flow itself. If we prescribe $\boldsymbol{\omega}$ instead, can we always recover $\boldsymbol{u}$? In general $\boldsymbol{u}$ is not unique, since $\boldsymbol{u}$ and $\boldsymbol{u}' := \boldsymbol{u} + \nabla\phi$ will have the same vorticity field. The arbitrary component $\nabla\phi$ is called a *potential flow* with its defining characteristic being that it has *no vorticity*. However, if we assume the flow is incompressible we can find a unique solution subject to given boundary conditions.

Proposition 1.6 (Biot-Savart integral)**.** *For a given $\boldsymbol{\omega}$ on a bounded domain D with $\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ on ∂D, there exists a unique incompressible velocity field with $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on ∂D, given by*

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_v(\boldsymbol{x}) + \nabla\phi,$$

where

$$\boldsymbol{u}_v(\boldsymbol{x}) = \frac{1}{4\pi}\int_D \frac{(\boldsymbol{x}' - \boldsymbol{x})\times\boldsymbol{\omega}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,\mathrm{d}V',$$

with ϕ determined by

$$\Delta\phi = 0, \qquad \boldsymbol{n}\cdot(\nabla\phi + \boldsymbol{u}_v) = 0 \text{ on } \partial D.$$

▶ $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ is required for this to be a physical flow, i.e. no flow through the walls. Here $\boldsymbol{u}_v$ carries all the vorticity of the flow, whereas the potential flow $\nabla\phi$ is there to enforce the no flow through boundary condition.

Proof. Since $\Delta\phi = 0$ we have that $\nabla\cdot\boldsymbol{u}_v = 0$ so we can write $\boldsymbol{u}_v = \nabla\times\boldsymbol{a}$. If we assume $\nabla\cdot\boldsymbol{a} = 0$, then we need

$$\boldsymbol{\omega} = \nabla\times\nabla\times\boldsymbol{a} = -\Delta\boldsymbol{a}. \tag{1.31}$$

This is Poisson's equation and the solution in three dimensions is [this is derived in the PDEs course]

$$\boldsymbol{a}(\boldsymbol{x}) = \frac{1}{4\pi}\int_D \frac{\boldsymbol{\omega}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\mathrm{d}V'. \tag{1.32}$$

This solution satisfies $\nabla \cdot \boldsymbol{a} = 0$ provided $\boldsymbol{\omega} \cdot \boldsymbol{n} = 0$ on ∂D [problem sheet]. Then

$$\boldsymbol{u}_v = \nabla \times \boldsymbol{a} = \frac{1}{4\pi} \int_D \nabla \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} \times \boldsymbol{\omega}(\boldsymbol{x}') \, \mathrm{d}V'. \tag{1.33}$$

Noting that

$$\nabla \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = -\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|^2} \nabla |\boldsymbol{x} - \boldsymbol{x}'| = \frac{\boldsymbol{x}' - \boldsymbol{x}}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \tag{1.34}$$

gives the result. □

▶ In practice we will usually consider infinite or semi-infinite domains, in which case ϕ is determined by the flow's behaviour at infinity.

Example → Potential flow between two walls.
Consider 2 walls at $y = \pm h$. Flow enters the gap between the walls at $x = -L$ with $\boldsymbol{u} = U_0 \boldsymbol{e}_x$, and leaves the gap at $x = L$ with $\boldsymbol{u} = U_0 \boldsymbol{e}_x$. Assuming $\boldsymbol{\omega} = 0$ between the walls we have that

$$\boldsymbol{u} = \nabla \phi, \quad \text{where} \quad \Delta \phi = 0.$$

On the walls ($y = \pm h$) $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ implies that $\frac{\partial \phi}{\partial y} = 0$. We also have the constraint that $\frac{\partial \phi}{\partial x} = U_0$ on $x = \pm L$. By inspection we can see that a solution for ϕ satisfying these boundary conditions and $\Delta \phi = 0$ is

$$\phi = U_0 x + c,$$

showing that the potential flow is simply $\boldsymbol{u} = U_0 \boldsymbol{e}_x$ uniformly between the walls.

▶ Note how in contrast to the stream function, lines of constant ϕ in a potential flow are perpendicular to the flow. This can be seen from the fact that $\boldsymbol{u} \times \nabla \phi = \boldsymbol{u} \times \boldsymbol{u} = 0$.

Example → Line vortex $\boldsymbol{\omega} = 2\pi k \delta(x)\delta(y)\boldsymbol{e}_z$ in an infinite domain with $|\boldsymbol{u}| \to 0$ as $|\boldsymbol{x}| \to \infty$.
The behaviour at infinity implies we should set $\phi = 0$. Due to the δ-functions, the Biot-Savart integral reduces to a line integral in z' along $x' = y' = 0$. In cylindrical coordinates (r, θ, z) we have $\boldsymbol{x}' - \boldsymbol{x} = -r\boldsymbol{e}_r + (z' - z)\boldsymbol{e}_z$, so

$$\boldsymbol{u} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{(-r\boldsymbol{e}_r) \times (2\pi k \boldsymbol{e}_z)}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, \mathrm{d}z' = \frac{kr\boldsymbol{e}_\theta}{2} \int_{-\infty}^{\infty} \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, \mathrm{d}z' = \frac{kr\boldsymbol{e}_\theta}{2} \int_{-\infty}^{\infty} \frac{1}{\left[r^2 + (z - z')^2\right]^{\frac{3}{2}}} \, \mathrm{d}z'.$$

Using the indefinite integral

$$\int \frac{\mathrm{d}x}{(a^2 + x^2)^{\frac{3}{2}}} = \frac{x}{a^2 \sqrt{a^2 + x^2}},$$

we find

$$\boldsymbol{u} = \frac{kr\boldsymbol{e}_\theta}{2} \left[\frac{-(z - z')}{r^2 \sqrt{r^2 + (z - z')^2}} \right]_{-\infty}^{\infty} = \frac{kr\boldsymbol{e}_\theta}{2} \left(\frac{2}{r^2} \right) = \frac{k}{r} \boldsymbol{e}_\theta,$$

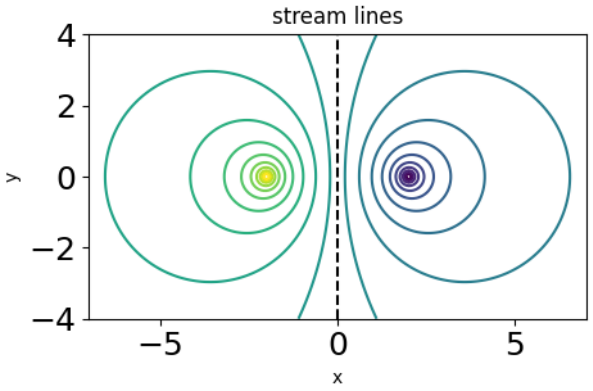
in agreement with our original velocity field.

Since the Bio-Savart integral is linear in $\boldsymbol{w}$, it has the useful property that solutions can be constructed through superposition. For line vortices, we can use this to find the potential flow that sastifies the $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ using the *method of images*.

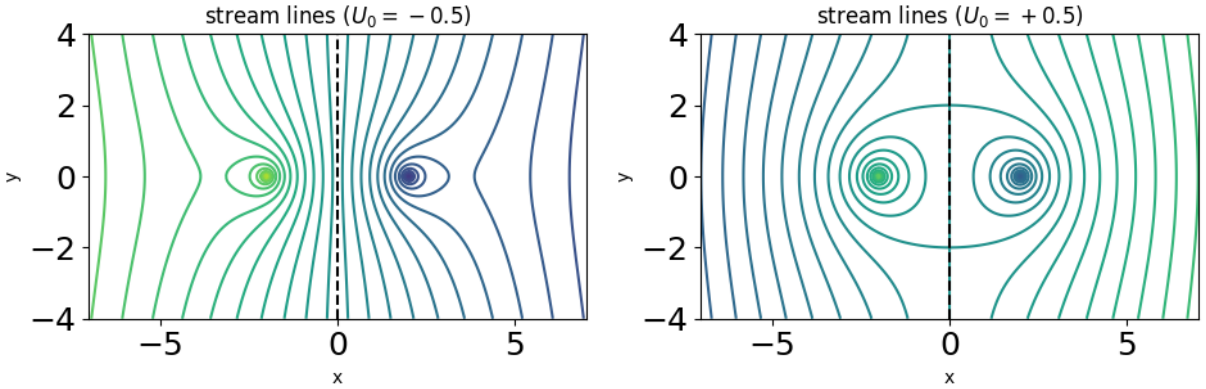
Example → Line vortex centered at $(x_0, 0)$ when $D = \{\boldsymbol{x} \in \mathbb{R}^2 : 0 \leq x < \infty, -\infty \leq y \leq \infty\}$.
Adding an image vortex at $(-x_0, 0)$ with opposite strength automatically satifies $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $x = 0$. The total vorticity within the infinite volume is then given by

$$\boldsymbol{\omega} = 2\pi k \delta(x - x_0)\delta(y)\boldsymbol{e}_z - 2\pi k \delta(x + x_0)\delta(y)\boldsymbol{e}_z.$$

The Bio-Savart integral can be split (homework question) into integrals to find the flow associated with each line vortex separately, where both integrals are evaluated over the infinite domain. The total flow is a superposition of each rotational flow. When $x_0 = 2$, the streamlines in the xy-plane look like:



Example → Addition of a uniform flow.
If instead of decaying, the flow became uniform and parallel to the wall at $|\boldsymbol{x}| \to \infty$ (still satisfying $\boldsymbol{u} \cdot \boldsymbol{n} = 0$) we could model this by adding a uniform potential flow $\boldsymbol{u} = U_0 \boldsymbol{e}_y$ corresponding to $\phi = U_0 x$. Two realisations of this flow are shown below.



1.9 Integrals over material domains

In addition to the usual Divergence and Stokes' theorems from AMV, we will need the following formulae for time derivatives of integrals over material domains in the sections to follow.

Proposition 1.7 (Transport Theorem)**.** *For any scalar function $f(\boldsymbol{x}, t)$ and any material volume D_t,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{D_t} f \ \mathrm{d}V = \int_{D_t} \left(\frac{\mathrm{D}f}{\mathrm{D}t} + f \nabla \cdot \boldsymbol{u} \right) \mathrm{d}V.$$

Proof. The trick for differentiating an integral over a changing volume is to pull it back to an integral over the fixed volume D_0 at $t = 0$, so that we can take the derivative inside the integral.

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} f\ \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} f\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{1.35}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_0} fJ\,\mathrm{d}a\mathrm{d}b\mathrm{d}c \tag{1.36}$$

$$= \int_{D_0} \frac{\mathrm{D}}{\mathrm{D}t}(fJ)\,\mathrm{d}a\mathrm{d}b\mathrm{d}c \tag{1.37}$$

$$= \int_{D_0} \left(\frac{\mathrm{D}f}{\mathrm{D}t}J + f\frac{\mathrm{D}J}{\mathrm{D}t}\right)\,\mathrm{d}a\mathrm{d}b\mathrm{d}c \tag{1.38}$$

$$= \int_{D_0} \left(\frac{\mathrm{D}f}{\mathrm{D}t} + f\nabla\cdot\boldsymbol{u}\right) J\,\mathrm{d}a\mathrm{d}b\mathrm{d}c, \tag{1.39}$$

using Lemma 1.3 in the last step. Changing coordinates back from $\boldsymbol{a}$ to $\boldsymbol{x}$ gives the result. □

Proposition 1.8. *For any vector field $\boldsymbol{v}(\boldsymbol{x},t)$ and any material curve γ_t,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\gamma_t} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{l} = \int_{\gamma_t}\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \boldsymbol{v}\cdot(\nabla\boldsymbol{u})\right)\cdot\mathrm{d}\boldsymbol{l}.$$

Proof. To derive this formula we use a similar trick as for the volume integral and map the integral back to $t=0$ using the Jacobian matrix: $dl_i = J_{ij}da_j = \frac{\partial x_i}{\partial a_j}da_j$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\gamma_t} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{l} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\gamma_t} v_i\,\mathrm{d}x_i \tag{1.40}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\gamma_0} v_i\frac{\partial x_i}{\partial a_j}\,\mathrm{d}a_j \tag{1.41}$$

$$= \int_{\gamma_0}\left[\frac{\mathrm{D}v_i}{\mathrm{D}t}\frac{\partial x_i}{\partial a_j} + v_i\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x_i}{\partial a_j}\right)\right]\,\mathrm{d}a_j \tag{1.42}$$

$$= \int_{\gamma_0}\left[\frac{\mathrm{D}v_i}{\mathrm{D}t}\frac{\partial x_i}{\partial a_j} + v_i\frac{\partial u_i}{\partial a_j}\right]\,\mathrm{d}a_j \tag{1.43}$$

$$= \int_{\gamma_0}\left[\frac{\mathrm{D}v_i}{\mathrm{D}t}\frac{\partial x_i}{\partial a_j} + v_i\frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial a_j}\right]\,\mathrm{d}a_j \tag{1.44}$$

$$= \int_{\gamma_t}\frac{\mathrm{D}v_i}{\mathrm{D}t}\,\mathrm{d}x_i + \int_{\gamma_t} v_i\frac{\partial u_i}{\partial x_k}\,\mathrm{d}x_k \tag{1.45}$$

$$= \int_{\gamma_t}\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \boldsymbol{v}\cdot(\nabla\boldsymbol{u})\right)\cdot\mathrm{d}\boldsymbol{l}. \tag{1.46}$$

□

2 Dynamics of ideal fluids

Now we will derive the dynamical equations that determine the velocity field $\boldsymbol{u}$. To do this, we will apply two basic physical laws to a fluid element D_t:

1. *Conservation of mass.*
2. *Conservation of momentum* (Newton's second law).

The basic forms of these two conservation laws are

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t}\rho\,\mathrm{d}V = 0, \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t}\rho\boldsymbol{u}\,\mathrm{d}V = \int_{D_t}\boldsymbol{F}\,\mathrm{d}V. \tag{2.2}$$

Here $\rho(\boldsymbol{x},t)$ is a scalar field describing the *mass density*, or mass per unit volume, and $\boldsymbol{F}$ is a vector field describing the force per unit volume acting on the fluid. In the following sections, we will show that these integral conservation laws are equivalent to partial differential equations (PDEs): the governing equations of fluid mechanics.

▶ We will usually assume ρ and $\boldsymbol{F}$ to be smooth on the macroscopic scale.

2.1 Conservation of mass

To see that (2.1) is equivalent to a PDE, apply the transport theorem (Proposition 1.7) to write (2.1) as

$$\int_{D_t}\left(\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\boldsymbol{u}\right)\,\mathrm{d}V = 0. \tag{2.3}$$

Now this applies for any material volume D_t, but the only way this is possible (assuming smoothness of the integrand) is if ρ satisfies everywhere the PDE

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho\nabla\cdot\boldsymbol{u}=0 \qquad\Longleftrightarrow\qquad \boxed{\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0.} \tag{2.4}$$

This is called the *(mass) continuity equation.*

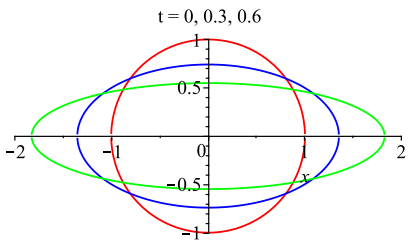
Example → Stagnation point flow $\boldsymbol{u}(x,y,t)=(x,-y)$ with initial density $\rho(x,y,0)=x^2+y^2$. This flow is incompressible since $\nabla\cdot\boldsymbol{u}=1-1=0$. Hence ρ satisfies

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}=0,$$

meaning that it is a Lagrangian invariant. In Lagrangian form $\rho(a,b,t)=a^2+b^2$. To find ρ as a function of x,y,t, recall that we found the Lagrangian coordinates of this flow to be $\boldsymbol{x}=(a\mathrm{e}^t,b\mathrm{e}^{-t})$. Hence

$$\rho(x,y,t)=(x\mathrm{e}^{-t})^2+(y\mathrm{e}^t)^2=x^2\mathrm{e}^{-2t}+y^2\mathrm{e}^{2t}.$$

The contours of constant density are initially circles centred at the origin, but are flattened into ellipses by the flow.



2.2 Conservation of momentum for an ideal fluid

To derive the PDE form of (2.2), we first need to know the force $\boldsymbol{F}$ acting on the fluid.

In general, we assume that the force on a fluid element D_t is

$$\int_{D_t} \boldsymbol{F}\,\mathrm{d}V = \int_{D_t} \rho \boldsymbol{f}\,\mathrm{d}V + \oint_{\partial D_t} \boldsymbol{\sigma}\cdot\,\mathrm{d}\boldsymbol{S}, \tag{2.5}$$

where

1. $\boldsymbol{f}$ represents any external *body force* per unit mass, acting on the fluid inside D_t.
2. $\boldsymbol{\sigma}$ is the force per unit area, or *stress*, exerted on the boundary of D_t by the surrounding fluid.

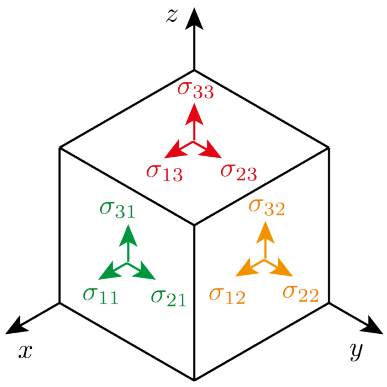
▶ Body forces are familiar from particle dynamics, but stress is a new concept.

Example → A common body force is gravity, $\boldsymbol{f} = -g\boldsymbol{e}_z$. Other examples might be magnetic forces if the fluid conducts electricity, or inertial "forces" such as the Coriolis and centrifugal forces when the fluid is in a rotating reference frame (discussed later). In $\boldsymbol{f} = 0$ we say the system is *unforced*.

In order that $\boldsymbol{\sigma}\cdot\,\mathrm{d}\boldsymbol{S}$ be a vector, $\boldsymbol{\sigma}$ must be a rank-two tensor, called the *stress tensor*. At any point, it has 9 components:

$$\boldsymbol{\sigma} = (\sigma_{ij}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}. \tag{2.6}$$

Physically, the index i of σ_{ij} gives the coordinate direction in which the force acts, and the index j gives the orientation of the surface it is acting on. To find the stress on a surface of arbitrary orientation $\boldsymbol{n}$, take $\boldsymbol{\sigma}\cdot\boldsymbol{n}$, which gives a vector with components $\sigma_{ij}n_j$.



Example → $\boldsymbol{n} = \boldsymbol{e}_x$.

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix}.$$

To specify $\boldsymbol{\sigma}$, we need a model for how our particular fluid behaves. This term, we will consider only an *ideal* (also known as *inviscid*) fluid, meaning that we neglect any internal friction (*viscosity*) between neighbouring fluid elements.

Proposition 2.1. *The stress tensor for an ideal fluid takes the form* $\sigma_{ij} = -p\delta_{ij}$.

Proof. Since there is no internal friction, an ideal fluid can have no tangential/shear stresses, only normal stresses. Thus it must be that

$$\boldsymbol{\sigma} = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix}. \tag{2.7}$$

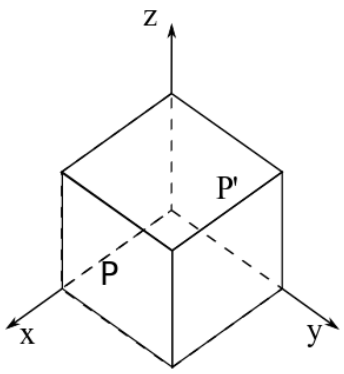
However, this only accounts for shear stresses on surfaces aligned with the coordinate axes. To avoid shear stresses on surfaces of any arbitrary orientation $\boldsymbol{n}$, we need

$$(\boldsymbol{\sigma}\cdot\boldsymbol{n})\times\boldsymbol{n} = 0 \quad \Longrightarrow \quad \begin{cases} (p_2-p_3)n_2n_3 = 0, \\ (p_3-p_1)n_3n_1 = 0, \\ (p_1-p_2)n_1n_2 = 0 \end{cases} \quad \Longrightarrow \quad p_1 = p_2 = p_3 := -p. \tag{2.8}$$

□

The scalar field $p(\boldsymbol{x}, t)$ is called the *pressure*. The minus sign ensures that positive p corresponds to a force pushing in on a fluid particle, as intuition suggests.

Example → Consider a fluid element aligned to the axes as shown with sides of length d.



Pressure from the surroundings varies smoothly in the x-direction from P on one side of the element to P' on the other. There is no net force in the y or z directions as the pressure on each opposing face is balanced. Therefore, the total net force on the fluid element is:

$$\oint_{\partial D_t} \boldsymbol{\sigma}\cdot\,\mathrm{d}\boldsymbol{S} = -\int P\boldsymbol{e}_x dS + \int P'\boldsymbol{e}_x dS = (P'-P)d^2\boldsymbol{e}_x.$$

i.e. if $P' > P$ the fluid element will feel a net force in the positive x direction as we expect.

Example → Barrel implosion from atmospheric pressure.
`https://www.youtube.com/watch?v=JsoE4F2Pb20`

Now we can derive the PDE form of (2.2) for an ideal fluid. First consider the right-hand side, which gives

$$\int_{D_t} \boldsymbol{F}\,\mathrm{d}V = \int_{D_t} \rho\boldsymbol{f}\,\mathrm{d}V + \oint_{\partial D_t} \boldsymbol{\sigma}\cdot\,\mathrm{d}\boldsymbol{S}, \tag{2.9}$$

$$= \int_{D_t} \left(\rho\boldsymbol{f} + \nabla\cdot\boldsymbol{\sigma}^\top\right)\mathrm{d}V \quad \text{(by the divergence theorem)}, \tag{2.10}$$

$$= \int_{D_t} \left(\rho\boldsymbol{f} - \nabla p\right)\mathrm{d}V. \tag{2.11}$$

where for the second and third lines we have used that $\nabla \cdot \boldsymbol{\sigma}^\top$ is a vector with components $(\nabla \cdot \boldsymbol{\sigma}^\top)_i = \dfrac{\partial \sigma_{ij}}{\partial x_j}$. Now apply the transport theorem to the left-hand side:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{D_t} \rho \boldsymbol{u}\, \mathrm{d}V = \int_{D_t} \left(\frac{\mathrm{D}(\rho \boldsymbol{u})}{\mathrm{D}t} + \rho \boldsymbol{u} \nabla \cdot \boldsymbol{u} \right) \mathrm{d}V, \tag{2.12}$$

$$= \int_{D_t} \left(\frac{\mathrm{D}\rho}{\mathrm{D}t}\boldsymbol{u} + \rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \rho \boldsymbol{u} \nabla \cdot \boldsymbol{u} \right) \mathrm{d}V, \tag{2.13}$$

$$= \int_{D_t} \left((-\rho \nabla \cdot \boldsymbol{u})\boldsymbol{u} + \rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \rho \boldsymbol{u} \nabla \cdot \boldsymbol{u} \right) \mathrm{d}V \qquad \text{(by mass continuity)}, \tag{2.14}$$

$$= \int_{D_t} \rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\, \mathrm{d}V. \tag{2.15}$$

Putting these two equations together gives

$$\int_{D_t} \left(\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} - \rho \boldsymbol{f} + \nabla p \right) \mathrm{d}V = 0. \tag{2.16}$$

Once again, assuming smoothness of the integrand, the only way this is possible for an arbitrary domain D_t is if ρ, $\boldsymbol{u}$ and p obey the PDE

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \rho \boldsymbol{f} - \nabla p \qquad \Longleftrightarrow \qquad \boxed{\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} - \frac{1}{\rho}\nabla p.} \tag{2.17}$$

This is called the *momentum equation*.

2.3 Incompressible Euler equations

Since the body force $\boldsymbol{f}$ is imposed, Equations (2.4) and (2.17) constitute a set of four equations, but we have five unknowns ρ, u_1, u_2, u_3, and p. We therefore need a fifth equation to define a complete system. This must come from additional physics.

The simplest choice is to assume an *incompressible fluid* with uniform, constant density $\rho = \rho_0$. This is a reasonable assumption for liquids but not always for gases.

For $\rho = \rho_0$, the continuity equation simplifies and we have the system

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.18}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} - \frac{1}{\rho_0}\nabla p, \tag{2.19}$$

called the *incompressible Euler equations*. There are still four equations but now only four unknowns, u_1, u_2, u_3 and p.

What about boundary conditions? Since *ideal* fluids (incompressible or not) feel no friction from the boundaries, fluid is free to flow tangential to these boundaries. Therefore, we can only impose that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, i.e. that there is no flow through the boundaries.

First let us look at some simple steady solutions (meaning $\partial \boldsymbol{u}/\partial t = 0$).

Example → Shear flow $\boldsymbol{u} = G(y)\boldsymbol{e}_x$ in a channel $0 < y < 1$.
Clearly this satisfies $\nabla \cdot \boldsymbol{u} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$. Moreover,

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z} \right)(G\boldsymbol{e}_x) = G \frac{\partial G}{\partial x}\boldsymbol{e}_x = 0, \qquad \frac{\partial \boldsymbol{u}}{\partial t} = 0.$$

Taking $p = p_0$ (any constant) we find that this flow is a solution of the incompressible Euler equations with $\boldsymbol{f} = \boldsymbol{0}$.

Example → Cup of tea $V = \{\boldsymbol{x} \in \mathbb{R}^3 : x^2 + y^2 < R^2, 0 < z < H\}$, with gravity $\boldsymbol{f} = -g\boldsymbol{e}_z$. To make things realistic, we need to impose some boundary conditions:

$$\text{At } z = 0 \text{ (bottom of cup)}, \ \boldsymbol{n} = -\boldsymbol{e}_z \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = 0.$$
$$\text{At } r = R \text{ (side of cup)}, \ \boldsymbol{n} = \boldsymbol{e}_r \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = 0.$$

We could allow for motion of the top surface (see later), but for simplicity lets assume that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ there also. 1. One possible solution is static $\boldsymbol{u} = \boldsymbol{0}$. Then we trivially have $\nabla \cdot \boldsymbol{u} = 0$ and the momentum equation reduces to

$$-g - \frac{1}{\rho_0}\frac{\partial p}{\partial z} = 0 \qquad \Longrightarrow \qquad p(z) = \rho_0 g(H - z).$$

This is called *hydrostatic equilibrium*. Here I chose the arbitrary constant to give $p = 0$ on the surface of the liquid. Note that the pressure increases for smaller z, because there is a greater weight of tea pushing down from above.

2. Another possible solution is an axisymmetric flow, $\boldsymbol{u} = G(r)\boldsymbol{e}_\theta$ in cylindrical coordinates (r, θ, z). We can see that the continuity equation is satisfied if we use the correct expression for divergence in cylindrical coordinates:

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = \frac{1}{r}\frac{\partial G}{\partial \theta} = 0.$$

For the momentum equation, we need to compute $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ in cylindrical coordinates:

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \left(u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z}\right)(u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta + u_z \boldsymbol{e}_z) = \frac{G}{r}\frac{\partial (G\boldsymbol{e}_\theta)}{\partial \theta} = \frac{G^2}{r}\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\frac{G^2}{r}\boldsymbol{e}_r.$$

So the momentum equation becomes

$$-\frac{G^2}{r}\boldsymbol{e}_r = -g\boldsymbol{e}_z - \frac{1}{\rho_0}\nabla p.$$

It is convenient to write the pressure as $p(\boldsymbol{x}) = p_s(z) + p_d(\boldsymbol{x})$ (sometimes called *static* and *dynamic* pressure), where $p_s(z) = \rho_0 g(H - z)$. Then $p_d = p_d(r)$ and the momentum equation is satisfied providing

$$\frac{G^2}{r} = \frac{1}{\rho_0}\frac{\mathrm{d}p_d}{\mathrm{d}r} \qquad \Longrightarrow \qquad p_d(r) = \int_0^r \rho_0 \frac{G(s)^2}{s}\,\mathrm{d}s.$$

► In general, given a known velocity $\boldsymbol{u}$ we can find the corresponding p by taking the divergence of the momentum equation, which leads to

$$\Delta p = \rho_0 \nabla \cdot \big(\boldsymbol{f} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\big).$$

This is a Poisson equation which, with the addition of boundary conditions $\boldsymbol{n} \cdot \nabla p = 0$, determines p. Since it is an elliptic PDE, a change of $\boldsymbol{u}$ at one location instantly affects the pressure, and therefore the acceleration of the fluid everywhere else. In this sense, ideal fluids "violate special relativity".

2.4 Energy

The incompressible Euler equations admit several conserved quantities.

Proposition 2.2. *In a fixed volume (V) with $\boldsymbol{u}\cdot\boldsymbol{n}=0$, the incompressible Euler equations with a conservative body force $\boldsymbol{f}=-\nabla U$ conserve kinetic energy,*

$$\frac{\mathrm{d}E}{\mathrm{d}t}=0 \qquad \textit{where} \quad E=\tfrac{1}{2}\int_V \rho_0|\boldsymbol{u}|^2\,\mathrm{d}V.$$

Proof. For simplicity, set $\rho_0=1$ (without loss of generality). Then

$$\frac{\mathrm{d}E}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\int_V \tfrac{1}{2}|\boldsymbol{u}|^2\,\mathrm{d}V \tag{2.20}$$

$$=\int_V \tfrac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{u}\cdot\boldsymbol{u})\,\mathrm{d}V \tag{2.21}$$

$$=\int_V \boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial t}\,\mathrm{d}V \tag{2.22}$$

$$=\int_V \boldsymbol{u}\cdot\Big(-\nabla U-\nabla p-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\Big)\,\mathrm{d}V. \tag{2.23}$$

Now apply the vector identity $\boldsymbol{u}\cdot\nabla\boldsymbol{u}=(\nabla\times\boldsymbol{u})\times\boldsymbol{u}+\frac{1}{2}\nabla\big(|\boldsymbol{u}|^2\big)$ to find

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\int_V \boldsymbol{u}\cdot\nabla\Big(U+p+\tfrac{1}{2}|\boldsymbol{u}|^2\Big)\,\mathrm{d}V \tag{2.24}$$

Now use the identity $\boldsymbol{u}\cdot\nabla f=\nabla\cdot(f\boldsymbol{u})-f\nabla\cdot\boldsymbol{u}$, and the fact that $\nabla\cdot\boldsymbol{u}=0$, to see that

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-\oint_{\partial V}\Big(U+p+\tfrac{1}{2}|\boldsymbol{u}|^2\Big)\boldsymbol{u}\cdot\,\mathrm{d}\boldsymbol{S}. \tag{2.25}$$

This surface integral vanishes provided that $\boldsymbol{u}\cdot\,\mathrm{d}\boldsymbol{S}=0$ on ∂V. □

Conservation of kinetic energy makes physical sense, since an ideal incompressible fluid has no internal friction or heating to dissipate the energy.

If the flow is steady, we have a stronger result:

Proposition 2.3 (Bernoulli's theorem)**.** *In an incompressible steady flow of an ideal fluid, with conservative body force $\boldsymbol{f}=-\nabla U$, the Bernoulli function*

$$B(\boldsymbol{x},t)=\frac{1}{\rho_0}p+\tfrac{1}{2}|\boldsymbol{u}|^2+U$$

is constant along streamlines.

▶ This is really a statement about conservation of total energy ("internal" + kinetic + potential), per unit mass.

Proof. Using our vector identity, we can rewrite the momentum equation as

$$\frac{\partial\boldsymbol{u}}{\partial t}+(\nabla\times\boldsymbol{u})\times\boldsymbol{u}=-\nabla\left(\frac{1}{\rho_0}p+\tfrac{1}{2}|\boldsymbol{u}|^2+U\right). \tag{2.26}$$

By definition, $\partial\boldsymbol{u}/\partial t=0$ for a steady flow, so taking the dot product with $\boldsymbol{u}$ shows that

$$\boldsymbol{u}\cdot\nabla B=0. \tag{2.27}$$

This means that B is constant along any streamline. □

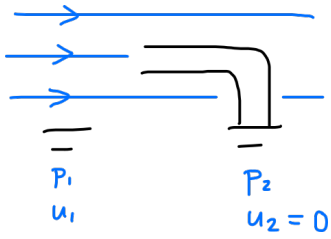
Finally, if both $\nabla \times \boldsymbol{u} = \boldsymbol{0}$ and $\nabla \cdot \boldsymbol{u} = 0$, there is a time-dependent version of Bernoulli's theorem.

Proposition 2.4 (Bernoulli's theorem for potential flow)**.** *In an irrotational, incompressible flow $\boldsymbol{u} = \nabla\phi$, with conservative body force $\boldsymbol{f} = -\nabla U$, there exists a function $C(t)$ such that*

$$\frac{\partial \phi}{\partial t} + \frac{1}{\rho_0} p + \frac{1}{2}|\nabla\phi|^2 + U = C(t).$$

Proof. See problem sheet. □

Example → Pitot tube.
This is a device used on aircraft to measure airspeed (the photo shows one on an Airbus A380).





Bernoulli's theorem tells us that

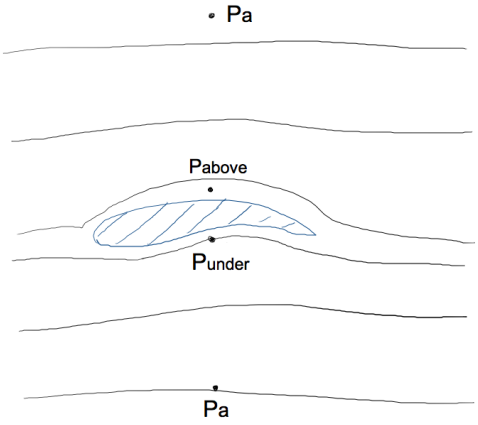
$$\tfrac{1}{2}u_1^2 + \frac{p_1}{\rho_0} = 0 + \frac{p_2}{\rho_0} \qquad \Longrightarrow \qquad u_1 = \sqrt{\frac{2(p_2 - p_1)}{\rho_0}}.$$

Example → Tornadoes.
Outside of the central eye, a tornado can be approximated by a steady irrotational flow (line vortex/Rankine vortex) where $\boldsymbol{u} = \frac{k}{r}\boldsymbol{e}_\theta$ for $r > r_{eye}$. In this flow $\nabla \times \boldsymbol{u} = \boldsymbol{0}$ and (2.26) implies

$$\nabla B = \boldsymbol{0} \quad \Longrightarrow \; B = \text{constant}$$

Even though flows at different radii are not connected by a streamline, they have the same value of B. Closer to the eye, $\boldsymbol{u}$ increases so p must decrease. This explains why tornadoes are associated with regions of low pressure. In fact, it can be shown that any curved streamline is associated with a local pressure drop around which the streamline curves.

Example → How does an aeroplane fly?
This is often attributed to Bernoulli's theorem, since the shape of the wing makes air flow faster above the wing than below it. Assuming Bernoulli's theorem is valid, this would produce a pressure drop above the wing. However, streamlines do not loop around a wing and the flow is not irrotational so we cannot assume B is constant everywhere. The reason for the pressure drop is a bit more subtle, and is related to the streamline curvature.



Consider a wing flying through undisturbed air at atmospheric pressure P_a. Moving towards the the underside of the wing from below the streamlines have an increasing downward curvature. P_{under} is on the "outside" of the curve so $P_{under} > P_a$. Conversely, moving towards the top of the wing from above the streamlines also increase in downward curvature, but now P_{above} is on the "inside" of the curve so $P_{above} < P_a$. Combined gives that $P_{above} < P_{below}$ = lift!

Example → Simple demonstration.
The curvature of streamlines also explains why blowing over a sheet of paper causes it to lift, but blowing past a straight sheet of paper produces no force.

▶ A nice explanation of Bernoulli in the context of wings can be found here: http://www3.eng.cam.ac.uk/outreach/Project-resources/Wind-turbine/howwingswork.pdf.

2.5 Vorticity dynamics

Vorticity is the key to many fluid phenomena, and understanding the evolution of the vorticity often holds the key to the entire evolution of the flow.

To derive the evolution equation for vorticity, take the curl of the incompressible (ideal) momentum equation (2.19) to find

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big) = \nabla \times \boldsymbol{f}. \tag{2.28}$$

If we assume a conservative body force $\boldsymbol{f} = -\nabla U$, then $\nabla \times \boldsymbol{f} = \boldsymbol{0}$. We now use a vector identity to see that

$$\nabla \times ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = \nabla \times \left[(\nabla\times\boldsymbol{u})\times\boldsymbol{u} + \tfrac{1}{2}\nabla(|\boldsymbol{u}|^2)\right] \tag{2.29}$$

$$= \nabla \times (\boldsymbol{\omega}\times\boldsymbol{u}). \tag{2.30}$$

Substituting into (2.28) gives

$$\boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega}\times\boldsymbol{u}) = \boldsymbol{0}.} \tag{2.31}$$

This is called the *vorticity equation*.

▶ The vorticity equation is a neat way of looking at the fluid because it eliminates the pressure p. As remarked earlier, we can always reconstruct p once we know $\boldsymbol{u}$.

▶ The vorticity equation is analogous to the momentum equation but describes the conservation of angular momentum.

We can also derive the Lagrangian form of this equation by using another vector identity

$$\nabla \times (\boldsymbol{\omega}\times\boldsymbol{u}) = \boldsymbol{\omega}\nabla\cdot\boldsymbol{u} - \boldsymbol{u}\nabla\cdot\boldsymbol{\omega} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} \tag{2.32}$$

$$= (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}, \tag{2.33}$$

where we used incompressibility. Thus

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} \quad \Longleftrightarrow \quad \boxed{\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u}.} \tag{2.34}$$

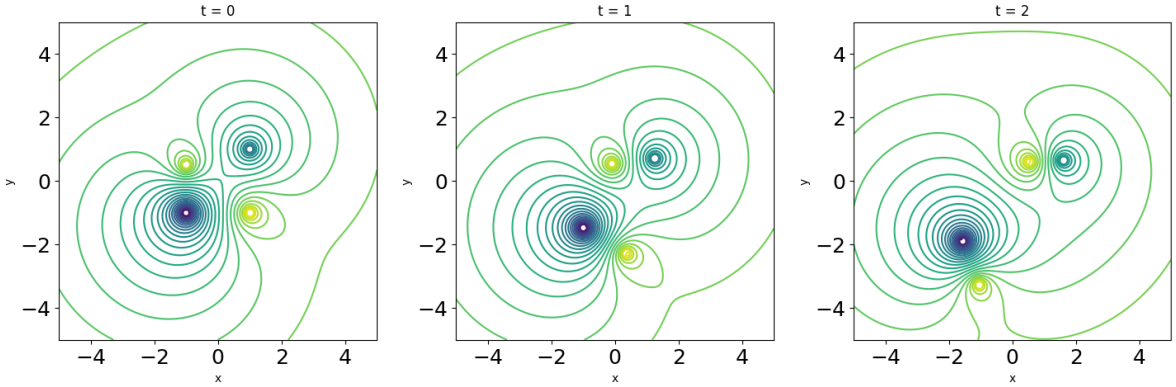
2.6 2D vorticity evolution

For a 2D time dependent flow $\boldsymbol{u} = \boldsymbol{u}(x, y, t)$ and $\boldsymbol{\omega} = \omega(x, y, t)\boldsymbol{e}_z$ where $\omega(x, y, t) = \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y}$. Therefore, $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$ vanishes and the vorticity equation reduces to

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = 0.$$

In other words, vorticity in a two-dimensional flow is a material scalar: the vorticity of each fluid particle is conserved and is simply advected by the flow.

Example → Line vortices revisited.
The above means that line vortices (point vortices in 2D) are advected by the overall flow and when multiple line vortices (or image vortices) are present they act to move each other around. Therefore, one can calculate the direction of motion of a given line vortex at a given position and time by considering the flow from all the other vorticies at this position. For a small number of vorticies this can be done analytically, but generally this requires a numerical treatment. This interaction leads to things like the self propulsion of vortex pairs, vortices travelling along walls and more. Below I've solved numerically the evolution of 4 asymmetric line vortices. Note how they group into two travelling pairs.



If the flow is steady so that $\partial\omega/\partial t = 0$, the vorticity equation further reduces to

$$\boldsymbol{u} \cdot \nabla\omega = 0.$$

Expressing $\boldsymbol{u}$ in terms of a stream function we find that

$$\omega = \boldsymbol{e}_z \cdot \boldsymbol{\omega} = \boldsymbol{e}_z \cdot \nabla \times \nabla \times (\psi \boldsymbol{e}_z) = -\Delta\psi. \tag{2.35}$$

The vorticity equation then becomes

$$\nabla \times (\psi \boldsymbol{e}_z) \cdot \nabla\omega = 0 \qquad \Longrightarrow \ \nabla\psi \times \boldsymbol{e}_z \cdot \nabla\omega = 0 \qquad \Longrightarrow \ \boldsymbol{e}_z \cdot \nabla\psi \times \nabla\omega = 0.$$

$\nabla\psi$ and $\nabla\omega$ are vectors normal to contours of ψ and ω, respectively. $\boldsymbol{e}_z \cdot \nabla\psi \times \nabla\omega = 0$ shows that these contours are parallel, which implies that ω is a function of ψ (and *vice-versa*). More precisely, $\boldsymbol{e}_z \cdot \nabla\psi \times \nabla\omega = \frac{\partial(\psi,\omega)}{\partial(x,y)}$, the Jacobian determinant of the transformation between (x, y) and (ψ, ω). A zero determinant implies that ω and ψ are not independent variables. Using (2.35) we can therefore write

$$\Delta\psi = F(\psi) \tag{2.36}$$

for some arbitrary function F. Therefore, the stream function satisfies Poisson's equation in steady, two-dimensional flows.

Example → Steady, two-dimensional vortex dipole.
The vortex dipole is a solution for the particular choice $F(\psi) = -k^2\psi$. Working in cylindrical coordinates (2.36) becomes

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)\right] = -k^2\psi.$$

If we assume a simple angular dependence and make the ansatz $\psi(r,\theta) = h(r)\sin\theta$, then

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}h}{\mathrm{d}r}\right)\sin\theta - \frac{h}{r^2}\sin\theta = -k^2 h\sin\theta$$
$$\iff \quad r^2\frac{\mathrm{d}^2h}{\mathrm{d}r^2} + r\frac{\mathrm{d}h}{\mathrm{d}r} + \left(r^2k^2 - 1\right)h = 0.$$

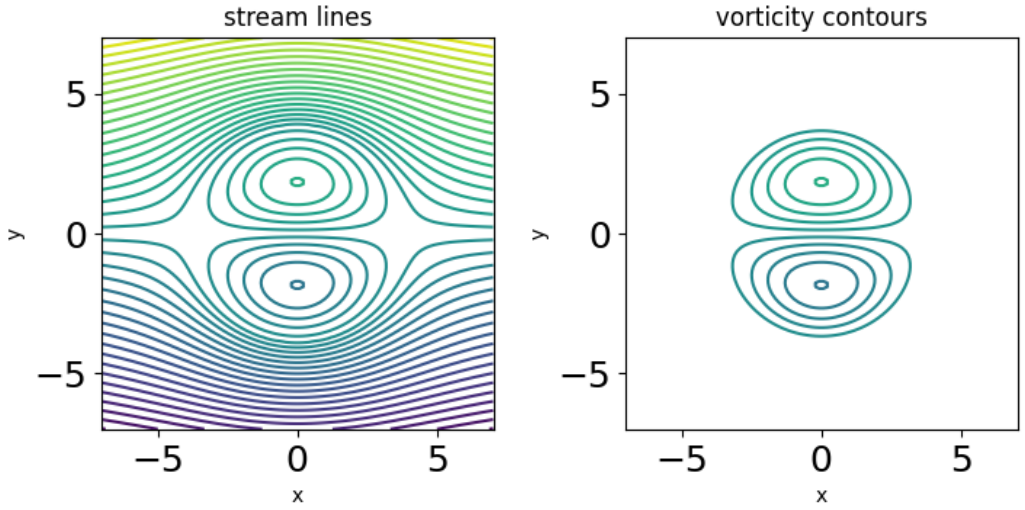
This is Bessel's ODE in the variable kr with $n = 1$. If we want the solution to be regular inside a disk $r < a$, we must take the solution $h(r) = cJ_1(kr)$, rather than $Y_1(kr)$. Thus

$$\psi(r,\theta) = cJ_1(kr)\sin\theta.$$

The corresponding vorticity is

$$\omega(r,\theta) = -\Delta\psi = k^2h(r)\sin\theta = ck^2J_1(kr)\sin\theta.$$

The two arbitrary constants c and k could be fixed by matching ω and/or $\boldsymbol{u}$ on the boundary $r = a$ to an exterior flow. Below is the solution when u_θ is matched to a potential flow that becomes uniform at $|\boldsymbol{x}| \to \infty$. Note how outside the disc $\omega = 0$.



2.7 3D vorticity evolution

In three dimensions, the vorticity equation tells us that $\boldsymbol{\omega}$ is not a material scalar but a *material vector field*, meaning that the vortex lines are material lines. This is easiest to see from the following expression.

Proposition 2.5 (Cauchy's formula)**.** *The solution to* (2.34) *may be written in Lagrangian variables as*

$$\omega_i(\boldsymbol{x}(\boldsymbol{a},t),t) = \frac{\partial x_i}{\partial a_j}\omega_j(\boldsymbol{a},0).$$

▶ This was derived by the mathematician Augustin Cauchy in 1815 in a prize-winning paper.

Proof. We make the ansatz $\omega_i = C_j \dfrac{\partial x_i}{\partial a_j}$ for some vector field $\mathbf{C}(\boldsymbol{x}, t)$. Taking the Lagrangian time derivative gives

$$\frac{\mathrm{D}\omega_i}{\mathrm{D}t} = \frac{\mathrm{D}C_j}{\mathrm{D}t}\frac{\partial x_i}{\partial a_j} + C_j \frac{\partial u_i}{\partial a_j}, \tag{2.37}$$

so substituting into the Lagrangian form of the vorticity equation gives

$$\frac{\mathrm{D}C_j}{\mathrm{D}t}\frac{\partial x_i}{\partial a_j} + C_j \frac{\partial u_i}{\partial a_j} = C_k \frac{\partial x_j}{\partial a_k}\frac{\partial u_i}{\partial x_j} = C_k \frac{\partial u_i}{\partial a_k}, \tag{2.38}$$

$$\implies \quad \frac{\mathrm{D}C_j}{\mathrm{D}t}\frac{\partial x_i}{\partial a_j} = 0. \tag{2.39}$$

In an incompressible flow we have $\det\left(\dfrac{\partial x_i}{\partial a_j}\right) = J = 1 \neq 0$, so we require $\dfrac{\mathrm{D}C_j}{\mathrm{D}t} = 0$. Now at time $t = 0$ we have $\boldsymbol{x} = \boldsymbol{a}$ so we must have $\omega_i(\boldsymbol{a}, 0) = C_j \frac{\partial a_i}{\partial a_j} = C_j \delta_{ij} = C_i$. □

Proposition 2.6 (Helmholtz's laws)**.** *In an ideal, incompressible fluid with conservative body force:*

H1. Fluid elements with no vorticity do not gain any as they move with the fluid.
H2. Vortex lines are material lines, and hence vortex tubes move with the fluid.
H3. The strength of a vortex tube is invariant as it evolves.

Proof of H1 & H2. Both results follow from Cauchy's formula. H1 is immediate. For H2, consider an initial vortex line element $\mathrm{d}\boldsymbol{a} = \boldsymbol{\omega}(\boldsymbol{a}, 0)\,\mathrm{d}l$. At time t, this evolves into a new line element $\mathrm{d}\boldsymbol{x}$ where

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial a_j}\mathrm{d}a_j = \frac{\partial x_i}{\partial a_j}\omega_j(\boldsymbol{a}, 0)\,\mathrm{d}l = \omega_i(\boldsymbol{x}, t)\,\mathrm{d}l. \tag{2.40}$$

This is still tangent to $\boldsymbol{\omega}$, so every vortex line must be a material curve. □

Proof of H3. Recall that the "strength" of a vortex tube is the circulation around any closed curve γ that encloses the tube. We will show later (Kelvin's theorem) that the circulation around any closed *material* curve γ_t is constant. Since a vortex tube moves with the fluid (H2) as does any material curve around the tube γ_t, it follows from Kelvin's theorem that the strength is invariant. □

H3 has important implications. If the diameter of a vortex tube is reduced, to maintain the overall circulation the vorticity within the tube must increase. This is known as *vortex stretching*, and the right hand side of equation (2.34) acts in this case as a source term. Whether vorticity can be increased to the point of singularity is a major unsolved problem (see next term).

▶ In the (translated) wording of Helmholtz's original paper, *each vortex line remains continually composed of the same water particles, while it swims forward with these water particles in the fluid.*

2.8 Helicity

Helmholtz's laws have significant implications for the evolution of ideal fluids. For example, if two vortex lines are initially linked, then this linkage must be preserved for all time as they move with the fluid.

▶ Helmholtz's work on vortex tubes famously led Lord Kelvin to propose that atoms are vortex knots in the ether. In his theory, different elements correspond to different mathematical knot types. Though wrong, this theory was instrumental in starting mathematical knot theory.

Example → Knotted vortex experiments [Kleckner & Irvine, *Nature Physics*, 2013].

The conservation of vortex line topology in Helmholtz's laws is captured in a global invariant called *helicity*.

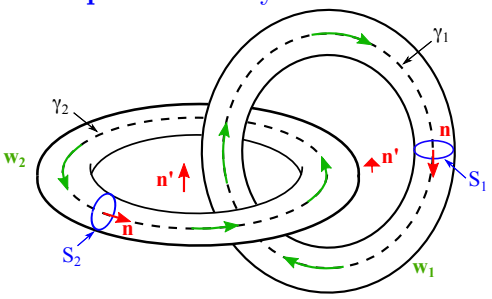
Proposition 2.7. *If* $\boldsymbol{\omega}\cdot\boldsymbol{n}=0$ *on* ∂V*, then the unforced incompressible Euler equations conserve helicity,*

$$\frac{\mathrm{d}H}{\mathrm{d}t}=0 \qquad \textit{where} \quad H=\int_V \boldsymbol{u}\cdot\boldsymbol{\omega}\,\mathrm{d}V.$$

Proof. This can be shown directly from the equations of motion with no reference to individual vortex lines – see the problem sheet. □

▶ Sometimes H is called *kinetic helicity* to distinguish it from other types of helicity, e.g. magnetic helicity in a conducting fluid.

Example → Helicity of two linked line vortices.



To see why H represents vortex line topology, consider a specific flow where $\boldsymbol{\omega}$ is confined to two closed, linked line vortices following the curves γ_1 and γ_2. Since $\boldsymbol{\omega}=\boldsymbol{0}$ away from these two curves, we can evaluate the integral by considering thin tubes surrounding around each curve. For each tube $dV = dl dS = \mathrm{d}\boldsymbol{l}\cdot\ \mathrm{d}\boldsymbol{S}$, where $\mathrm{d}\boldsymbol{l}=dl\,\boldsymbol{n}$ is the element of length along the tube and $\mathrm{d}\boldsymbol{S}=dS\,\boldsymbol{n}$ is its cross section.

Combining gives that for each tube

$$\boldsymbol{u}\cdot\boldsymbol{\omega}\,dV=(\boldsymbol{u}\cdot\boldsymbol{\omega})dl\,dS=(\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l})(\boldsymbol{\omega}\cdot\ \mathrm{d}\boldsymbol{S})$$

So that

$$H=\oint_{\gamma_1}\int_{S_1}(\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l})(\boldsymbol{\omega}\cdot\ \mathrm{d}\boldsymbol{S})+\oint_{\gamma_2}\int_{S_2}(\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l})(\boldsymbol{\omega}\cdot\ \mathrm{d}\boldsymbol{S})$$

Recall that the circulation around a closed curve is

$$C=\int_S \boldsymbol{\omega}\cdot\ \mathrm{d}\boldsymbol{S}$$

where S is the area enclosed by the curve. Applied to a vortex line, C gives the strength of the vortex which is invariant along its length. Therefore,

$$H=C_1\oint_{\gamma_1}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}+C_2\oint_{\gamma_2}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{l}$$

where C_1 and C_2 are the respective circulations/strengths. Now applying Stokes' theorem we see that

$$H = C_1 \int_{S_1'} \boldsymbol{\omega} \cdot \, \mathrm{d}\boldsymbol{S} + C_2 \int_{S_2'} \boldsymbol{\omega} \cdot \, \mathrm{d}\boldsymbol{S},$$

where S_1' and S_2' are the surfaces enclosed by curves γ_1 and γ_2. Due to their linking, the vorticity that threads the surface S_1' is $\boldsymbol{\omega}_2$ and similarly the vorticity threading S_2' is $\boldsymbol{\omega}_1$. Therefore each integral measures the circulation of the other curve, giving

$$H = C_1 C_2 + C_2 C_1 = 2C_1 C_2$$

From this we can see that if the two curves are not linked, they will have zero helicity. Moreover, we can write

$$H = 2\mathcal{L}_{12} C_1 C_2,$$

where $\mathcal{L}_{12}$ is the (Gauss) linking number of the two curves. In this case $L_{12} = 1$, but any number of links by the same or additional vortex rings are possible [problem sheet].

▶ Maybe you saw the linking number in Geometric Topology: you just add up the number of positive and negative crossings in any projection, and divide by two.

▶ In general, you can show that the helicity is the integral of the pairwise linking over all pairs of vortex lines in the fluid.

2.9 Circulation

Another way of looking at the materiality of vorticity in an ideal fluid is the following.

Proposition 2.8 (Kelvin's theorem)**.** *In an ideal, incompressible fluid with conservative body force, the circulation around a closed material curve γ_t is constant in time:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = 0.$$

Proof. Proposition 1.8 gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = \oint_{\gamma_t} \left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \boldsymbol{u} \cdot (\nabla \boldsymbol{u}) \right) \cdot \mathrm{d}\boldsymbol{l} \tag{2.41}$$

Using index notation,

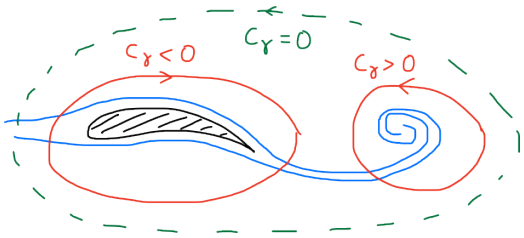
$$\oint_{\gamma_t} \boldsymbol{u} \cdot (\nabla \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{l} = \oint_{\gamma_t} u_i \frac{\partial u_i}{\partial x_j} \, \mathrm{d}l_j = \frac{1}{2} \oint_{\gamma_t} \frac{\partial}{\partial x_j} \Big(u_i u_i \Big) \, \mathrm{d}l_j = \frac{1}{2} \oint_{\gamma_t} \nabla |\boldsymbol{u}|^2 \cdot \mathrm{d}\boldsymbol{l}. \tag{2.42}$$

With a conservative body force $\boldsymbol{f} = -\nabla U$, the equation of motion (2.19) gives

$$\oint_{\gamma_t} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{l} = - \oint_{\gamma_t} \nabla \left(U + \frac{p}{\rho_0} \right) \cdot \mathrm{d}\boldsymbol{l}. \tag{2.43}$$

Both terms have been reduced to integrating gradients of continuous quantities around closed loops, so both vanish. □

Example → Starting vortex.
Since an aeroplane needs faster flow above its wing than below, there must be a negative circulation around its wing. Kelvin's theorem therefore implies that, for an aeroplane to start from rest (with $C_\gamma = 0$ around its wing) in ideal air, it must shed a *starting vortex* of opposite sign, so as to conserve circulation overall.



Aeroplanes do indeed shed starting vortices, although thanks to Helmholtz's law H1, their creation actually requires the presence of viscosity (non-idealness) near the surface of the wing.

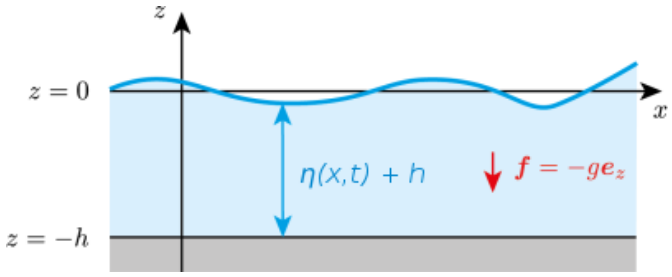
Example → Wing tip vortices.
Since vortex lines cannot end in mid-air, in 3D the starting vortex is connected to the ends of the wings by *wing tip vortices* forming a closed loop.



Wing tip vorticies can be visible when an aircraft passes through clouds at landing and take off (and are different from contrails created by the engines that produce cloud lines in the sky). Were air an ideal fluid these vortices would stretch across the globe! Instead, they viscously decay over a few minutes after a plane passes. Small planes can be flipped over by the wing tip vortices of large air liners, so pilots generally give each other a lot of space!

3 Water waves

An important class of solutions to the incompressible Euler equations are water waves. We will consider a layer of water $D_t = \{(x, y, z) : -h < z < \eta(x, y, t)\}$, so that the bed $z = -h$ is flat and the undisturbed water surface is $z = 0$. The free surface of the water is $z = \eta(x, y, t)$, which is now an unknown.



3.1 Free surface boundary conditions

Let $z = \eta(x, y, t)$ denote the free surface. As for the cup of tea earlier, we impose two boundary conditions at this surface:

1. $p = 0$ on $z = \eta(x, y, t)$.
2. $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ where $\boldsymbol{n}$ is the normal to $z = \eta(x, y, t)$.

▶ The first condition works because air is much less dense than water. It means that we are neglecting surface tension, and also the effect of the wind in creating waves.

To write the second condition in terms of η, note that it is equivalent to saying that fluid particles on the surface stay on the surface. In other words,

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{\mathrm{D}z}{\mathrm{D}t} \qquad \Longleftrightarrow \qquad \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta = u_z \quad \text{on } z = \eta. \tag{3.1}$$

This is called the *kinematic boundary condition*.

3.2 Governing equations

We will assume that the water is an ideal, incompressible, irrotational fluid, and that the only body force is gravity $\boldsymbol{f} = -g\boldsymbol{e}_z$.

Since both $\nabla \times \boldsymbol{u} = \boldsymbol{0}$ and $\nabla \cdot \boldsymbol{u} = 0$, we can write $\boldsymbol{u} = \nabla\phi$ for some velocity potential $\phi(x, y, z, t)$ satisfying

$$\boxed{\Delta\phi = 0 \qquad \text{in } -h < z < \eta.} \tag{3.2}$$

There are three boundary conditions:

1. $p = 0$ on $z = \eta$. We can eliminate p in terms of ϕ and η using Proposition 2.4, so our boundary condition may be written

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = C(t) \quad \text{on } z = \eta. \tag{3.3}$$

   In fact, we can just set $C(t) = 0$ by incorporating it into ϕ, since adding a uniform constant to ϕ does not change $\boldsymbol{u}$.
2. The kinematic condition (3.1), which gives

$$\frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta = \frac{\partial \phi}{\partial z} \quad \text{on } z = \eta. \tag{3.4}$$

3. The boundary condition $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $z = -h$, which gives

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on } z = -h. \tag{3.5}$$

3.3 Linearisation

The full system (3.2)-(3.5) is nonlinear and difficult to solve, but we can find approximate solutions by assuming that η and ϕ (along with their derivatives) are small and approximating equations (3.2)-(3.5) by *linearisation*.

Firstly, we neglect terms that are quadratic in ϕ and/or η, so the system becomes

$$\Delta\phi = 0 \quad \text{in } -h < z < \eta, \tag{3.6}$$

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{on } z = \eta, \tag{3.7}$$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{on } z = \eta, \tag{3.8}$$

$$\frac{\partial \phi}{\partial t} = 0 \quad \text{on } z = -h. \tag{3.9}$$

Secondly, we use Taylor series to expand the derivatives of ϕ at the free surface about the fixed plane $z = 0$, which gives

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=\eta} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0} + \eta \left.\frac{\partial^2\phi}{\partial z^2}\right|_{z=0} + \dots, \tag{3.10}$$

$$\left.\frac{\partial\phi}{\partial t}\right|_{z=\eta} = \left.\frac{\partial\phi}{\partial t}\right|_{z=0} + \eta \left.\frac{\partial^2\phi}{\partial z\partial t}\right|_{z=0} + \dots. \tag{3.11}$$

To be consistent with our neglect of quadratic and higher-order terms, we keep only the first terms of these expansions, so the linearised equations are

$$\Delta\phi = 0 \quad \text{in } -h < z < 0, \tag{3.12}$$

$$\frac{\partial\phi}{\partial t} + g\eta = 0 \quad \text{at } z = 0, \tag{3.13}$$

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial z} \quad \text{at } z = 0, \tag{3.14}$$

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{at } z = -h. \tag{3.15}$$

3.4 Travelling wave solutions

Firstly, let us look for separable *plane wave* solutions of the form

$$\phi(x, z, t) = X(x - ct)Z(z). \tag{3.16}$$

In other words, we look for waves moving at a constant *phase speed* in the x-direction. Substituting this into (3.12) gives

$$X''Z + XZ'' = 0 \qquad \Longrightarrow -\frac{X''}{X} = \frac{Z''}{Z} = k^2. \tag{3.17}$$

From boundary condition (3.15) we need $Z'(-h) = 0$, so the appropriate solutions are

$$Z(z) = \cosh\big[k(z + h)\big], \qquad X(x - ct) = \mathrm{Re}\big(\mathrm{e}^{ik(x-ct)}\big). \tag{3.18}$$

▶ Notice that the wave amplitude decreases exponentially with depth.

However, c and k are not independent, since we also need to satisfy (3.13) and (3.14). Combining those two conditions, we have

$$0 = \frac{\partial^2\phi}{\partial t} + g\frac{\partial\eta}{\partial t} = \frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} \qquad \text{on } z = 0. \tag{3.19}$$

Substituting our solution $\phi(x, z, t)$ into this condition gives

$$-c^2k^2\cosh(kh)X + gk\sinh(kh)X = 0, \tag{3.20}$$

and writing $\omega = ck$ shows that ω (or c) and k must satisfy the *dispersion relation*

$$\boxed{\omega^2 = gk\tanh(kh).} \tag{3.21}$$

This says that the phase speed of wave crests c depends on their wavenumber k (the spatial frequency). For this reason, water waves are called *dispersive*.

▶ By contrast, the waves on a stretched string are non-dispersive: their speed depends only on the tension in the string and its mass density.

▶ The wavelength of one of these waves is $\lambda = 2\pi/k$.

The solution for the free surface $\eta(x,t)$ is given by (3.13), namely

$$\eta(x,t) = -\frac{1}{g}\mathrm{Re}\left(\left.\frac{\partial\phi}{\partial t}\right|_{z=0}\right) = \mathrm{Re}\left(\frac{i\omega}{g}\phi(x,0,t)\right) = -\frac{\omega}{g}\cosh(kh)\,\mathrm{Im}\left(\mathrm{e}^{i(kx-\omega t)}\right). \tag{3.22}$$

▶ Those familiar with PDEs may be confused by the fact that a "wave" solution ϕ satisfies the Laplace equation $\Delta\phi = 0$ rather than the classical wave equation $\dfrac{\partial^2\phi}{\partial t^2} = c^2\Delta\phi$. But in fact the waves are in the free surface height – you can easily see from (3.22) that $\dfrac{\partial^2\eta}{\partial t^2} = c^2\dfrac{\partial^2\eta}{\partial x^2}$.

There are two important limiting cases:

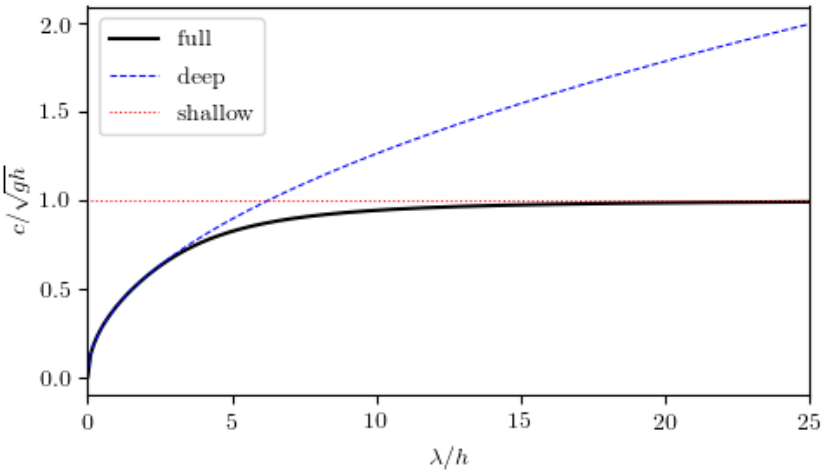
1. *Deep water,* $|kh| \gg 1$. As $k \to \infty$, the dispersion relation becomes
$$\omega^2 = gk\tanh(kh) \to gk. \tag{3.23}$$
So the wave speed $c \to \sqrt{g/k}$ becomes independent of h (the waves "don't feel the bottom").
2. *Shallow water/long wavelength,* $|kh| \to 0$. In this limit,
$$\omega^2 = gk\tanh(kh) \to gk^2h, \tag{3.24}$$
so these waves travel at a constant speed $c = \sqrt{gh}$.

Example → Ocean swell with wavelength $\lambda = 300\,\mathrm{m}$.
These are small-amplitude low-frequency waves generated by distant storms. The wavelength is small compared to an ocean depth of about 5 km away from continental margins. The deep water approximation suggests a frequency $\omega = \sqrt{gk} \approx 0.4\mathrm{s}^{-1}$.

A classic paper by Snodgrass et al. in 1966 (`https://doi.org/10.1098/rsta.1966.0022`) showed that ocean swell from storms in the Southern Ocean can propagate all the way across the Pacific to Alaska.

Example → Tsunami with wavelength 50 km.
The ocean depth is much less than the wavelength, so the shallow water approximation holds and $c = \sqrt{gh} \approx 220\,\mathrm{ms}^{-1}$. This is almost as fast as a subsonic passenger jet!

The following plot shows how the dispersion relation (3.21) matches to the deep (dashed blue) and shallow (dotted red) approximations:



▶ Notice how longer wavelengths travel faster, so it is the longest wavelengths that arrive first in the tsunami.

3.5 General solution

We can write a general solution as a superposition of plane wave solutions with different wavenumbers[1],

$$\phi(x,z,t) = \int_{-\infty}^{\infty} a(k)\cosh\left[k(z+h)\right] \mathrm{e}^{i(kx-\omega t)}\,\mathrm{d}k. \tag{3.25}$$

To find the coefficients $a(k)$, consider the initial condition

$$\phi(x,0,0) = \int_{-\infty}^{\infty} a(k)\cosh(kh)\,\mathrm{e}^{ikx}\,\mathrm{d}k. \tag{3.26}$$

Multiplying by e^{-ilx} and integrating over x gives

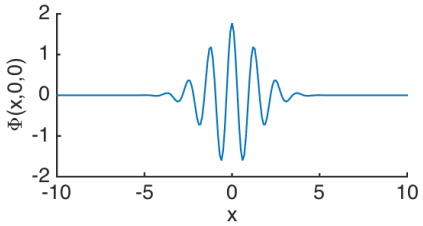
$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(x,0,0)\mathrm{e}^{-ilx}\,\mathrm{d}x &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(k)\cosh(kh)\mathrm{e}^{i(k-l)x}\,\mathrm{d}k\,\mathrm{d}x && (3.27)\\
&= \int_{-\infty}^{\infty} a(k)\cosh(kh)\left[\int_{-\infty}^{\infty}\mathrm{e}^{i(k-l)x}\,\mathrm{d}x\right]\mathrm{d}k && (3.28)\\
&= 2\pi\int_{-\infty}^{\infty} a(k)\cosh(kh)\delta(k-l)\,\mathrm{d}k && (3.29)\\
&= 2\pi a(l)\cosh(lh). && (3.30)
\end{aligned}$$

▶ In (3.29), we used the fact that $\int_{-\infty}^{\infty}\mathrm{e}^{ikx}\,\mathrm{d}x = 2\pi\delta(k)$. In other words, the Dirac δ-function is the Fourier transform of the function $f(x) = 1$.

If we specify a particular initial condition $\phi(x,0,0)$, then we can use this to compute the coefficients $a(k)$ in the general solution . It follows that

$$a(k) = \frac{1}{2\pi}\,\mathrm{sech}(kh)\int_{-\infty}^{\infty}\phi(x,0,0)\mathrm{e}^{-ikx}\,\mathrm{d}x. \tag{3.31}$$

Example → Gaussian wavepacket $\phi(x,0,0) = \sqrt{\frac{\pi}{\sigma}}\mathrm{e}^{-x^2/(4\sigma)}\mathrm{e}^{ik_0x}$.



We can calculate

$$a(k) = \frac{1}{2\pi}\mathrm{sech}(kh)\int_{-\infty}^{\infty}\phi(x,0,0)\mathrm{e}^{-ikx}\,\mathrm{d}x = \frac{1}{\sqrt{4\pi\sigma}}\mathrm{sech}(kh)\int_{-\infty}^{\infty}\mathrm{e}^{-x^2/(4\sigma)-i(k-k_0)x}\,\mathrm{d}x.$$

[1]Here we will leave it unwritten that the velocity field comes from only the *real part* of ϕ. This is a standard notational shorthand.

To do this integral, we need to write the integrand as e^{-u^2}. To do this, first complete the square

$$-\frac{x^2}{4\sigma} - i(k-k_0)x = -\frac{1}{4\sigma}\Big[x^2 + 4\sigma i(k-k_0)x\Big] = -\frac{1}{4\sigma}\Big[(x+2\sigma i(k-k_0))^2 + 4\sigma^2(k-k_0)^2\Big].$$

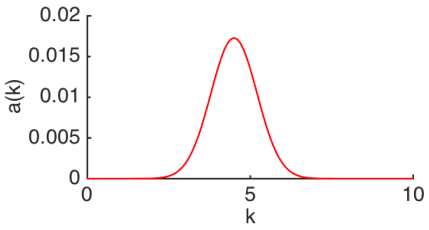
to give

$$a(k) = \frac{1}{\sqrt{4\pi\sigma}}\mathrm{sech}(kh)\mathrm{e}^{-\sigma(k-k_0)^2}\int_{-\infty}^{\infty}\mathrm{e}^{-\left[x+2\sigma i(k-k_0)\right]^2/(4\sigma)}\,\mathrm{d}x.$$

Then change variables to $u = \big[x + 2\sigma i(k-k_0)\big]/\sqrt{4\sigma}$ to give

$$a(k) = \frac{1}{\sqrt{\pi}}\mathrm{sech}(kh)\mathrm{e}^{-\sigma(k-k_0)^2}\int_{-\infty}^{\infty}\mathrm{e}^{-u^2}\,\mathrm{d}u = \mathrm{sech}(kh)\mathrm{e}^{-\sigma(k-k_0)^2}.$$

[Here we used the Gaussian integral $\int_{-\infty}^{\infty}\mathrm{e}^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$.]



Putting this back into (3.25) would lead to a nasty integral. However, since ϕ is only non-zero for k near k_0, we can Taylor expand

$$\omega(k) \approx \omega(k_0) + (k-k_0)c_\mathrm{G}, \qquad \text{where} \quad c_\mathrm{G} = \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k=k_0}.$$

The quantity c_G is called the *group velocity*. To see why, put this approximation into (3.25):

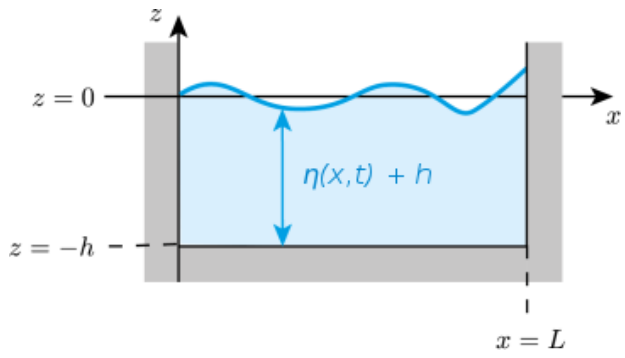
$$\begin{aligned}\phi(x,z,t) &\approx \int_{-\infty}^{\infty} a(k)\cosh\big[k(z+h)\big]\mathrm{e}^{ikx - i\omega(k_0)t - i(k-k_0)c_\mathrm{G}}\,\mathrm{d}k \\ &= \mathrm{e}^{i(k_0x-\omega(k_0)t)}\int_{-\infty}^{\infty} a(k)\cosh\big[k(z+h)\big]\mathrm{e}^{i(k-k_0)(x-c_\mathrm{G}t)}\,\mathrm{d}k.\end{aligned}$$

The factor in front represents a pure harmonic wave of wavenumber k_0, and the rest depends on x and t only through the combination $x - c_\mathrm{G}t$. So the envelope moves at the group velocity c_G.

▶ More generally, it may be shown that energy propagates at the group velocity $c_\mathrm{G}(k) = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$, which is calculated from the dispersion relation $\omega(k)$. For dispersive waves, c_G depends on k.

3.6 Standing waves

So far we have considered an infinite layer of water in x. Now consider a finite domain $D_t = \{(x,z) : -h < z < \eta(x,t), 0 < x < L\}$ with rigid walls $x = 0$ and $x = L$.



In such a domain, it does not make sense to look for travelling waves, so instead we look for standing waves of the form

$$\phi(x,z,t) = X(x)Z(z)\sin(\omega t). \tag{3.32}$$

In addition to (3.12)-(3.15), we need to impose $\boldsymbol{u}\cdot\boldsymbol{n} = \pm\dfrac{\partial\phi}{\partial x} = 0$ on the side walls.

Equation (3.12) separates in the same way as before, and again we find $Z(z) = \cosh\big[k(z+h)\big]$, but now the new boundary condition means that we must have

$$X(x) = \cos(kx) = \cos\left(\frac{n\pi x}{L}\right) \quad \text{with } n \in \mathbb{Z}. \tag{3.33}$$

The same dispersion relation (3.21) holds as for the infinite domain, but now k is "quantised" and

$$\omega^2 = \frac{gn\pi}{L}\tanh\left(\frac{n\pi h}{L}\right) \quad \text{for } n \in \mathbb{Z}. \tag{3.34}$$

So for a finite domain there is a discrete spectrum of possible wave frequencies, or *normal modes*.

▶ The mode number n tells you how many oscillations there are across the container. The lowest order mode, $n = 1$, is called the *fundamental mode*.

Example → Free surface.
The free surface height for a normal mode is again given by (3.13), so

$$\eta(x,t) = -\frac{1}{g}\frac{\partial\phi}{\partial t}\bigg|_{z=0} = -\frac{\omega}{g}\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi h}{L}\right)\cos(\omega t),$$

with ω given by (3.34).

▶ Although we started from a different ansatz to the infinite domain case, we could actually express a normal mode as a pair of counterpropagating travelling waves, bouncing off the walls of the container. Adding them up, they look like a standing wave. In particular,

$$\cos\left(\frac{n\pi x}{L}\right)\cos(\omega t) = \frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \omega t\right) + \cos\left(\frac{n\pi x}{L} + \omega t\right)\right].$$

This is why the dispersion relation for standing waves is the same as for travelling waves.

Just as the general solution in an infinite domain could be written as an integral over a continuous set of frequencies, the general solution in a container can be written as a sum over the discrete set of normal modes [see problem sheet].