KNOT CONCORDANCES AND ALTERNATING KNOTS

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Abstract. There is an infinitely generated free subgroup of the smooth knot concordance group with the property that no nontrivial element in this subgroup can be represented by an alternating knot. This subgroup has the further property that every element is represented by a topologically slice knot.

1. Introduction

Let $\mathcal{C}_a \subset \mathcal{C}$ denote the subgroup of the smooth three-dimensional knot concordance group generated by the set of all alternating knots. A simple construction reveals that connected sums of alternating knots, as well as their mirror images, are alternating; it follows that every element in $\mathcal{C}_a$ is represented by an alternating knot.

It has been known that $\mathcal{C}_a \neq \mathcal{C}$; in Section 2 we prove the following result, which also holds in the topological category.

Theorem 1. *For every positive integer N, there is an infinitely generated free subgroup $\mathcal{H}_N \subset \mathcal{C}/\mathcal{C}_a$ with the following property: If K represents a nontrivial class in $\mathcal{H}_N$, then any cobordism from K to an alternating knot has genus at least N.*

Our main result is this generalization.

Theorem 2. *For every positive integer N, there is an infinitely generated free subgroup $\mathcal{H}_N \subset \mathcal{C}/\mathcal{C}_a$ with the following properties.*

(1) *If K represents a nontrivial class in $\mathcal{H}_N$, then any cobordism from K to an alternating knot has genus at least N.*
(2) *Every class in $\mathcal{H}_N$ is represented by a topologically slice knot; in particular, the quotient $\mathcal{C}_{ts}/(\mathcal{C}_{ts} \cap \mathcal{C}_a)$ is infinitely generated, where $\mathcal{C}_{ts}$ is the concordance group of topologically slice knots.*

These results were inspired by three results concerning alternating knots. In [25], Ozsváth-Stipsicz-Szabó showed that the concordance invariant $\Upsilon_K(t)$, a piecewise linear function on $[0, 2]$, provides an obstruction to a knot being concordant to an alternating knot. In an earlier paper, Abe [1] showed that the difference between the Rasmussen invariant and the signature of a knot provides a bound on the *alternation number* of knot, $\mathrm{alt}(K)$, the minimum number of crossing changes required to convert K into an alternating knot. Extending these results, Feller-Pohlmann-Zentner [6] used $\Upsilon_K(t)$ to

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find another lower bound on the alternation number. The work here is built from the key observations of [1, 6].

For a knot K, let $\mathcal{A}_g(K)$ denote the minimum genus of a cobordism from K to an alternating knot and let $\mathcal{A}_s(K)$ denote the minimum number of double point singularities in a generically immersed concordance from K to an alternating knot. It is straightforward to show that both $\mathcal{A}_g$ and $\mathcal{A}_s$ induce functions from $\mathcal{C}/\mathcal{C}_a$ to the nonnegative integers.

The focus of [1, 6] is on bounding $\mathrm{alt}(K)$. Notice that a sequence of crossing changes on a knot yields a singular concordance, and its singularities can be resolved to produce an embedded surface. Thus, a lower bound on $\mathcal{A}_g(K)$ yields a bound on $\mathcal{A}_s(K)$ and hence on $\mathrm{alt}(K)$. The approach used in [1, 6] leads to much better bounds on $\mathrm{alt}(K)$ and $\mathcal{A}_s(K)$ than can be obtained via those on $\mathcal{A}_g(K)$. These will be presented in Section 6, after we conclude our work on the bounds on $\mathcal{A}_g$.

In the context of Heegaard Floer homology, alternating knots form a subgroup of the set of *quasi-alternating knots*. Using the results of [17], all our results extend to this setting; our theorems could be stated in terms of $\mathcal{C}_{qa}$, the subgroup of the concordance group generated by quasi-alternating knots, which, by [2, Lemma 2.3], consists precisely of the quasi-alternating knots.

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2. Algebraic invariants

We begin with an elementary result.

Lemma 3. *Let $\nu\colon \mathcal{C} \to \mathbb{R}$ be a homomorphism that vanishes on $\mathcal{C}_a$ and has the property that for all knots K, $g_4(K) \geq |\nu(K)|$. The genus of any cobordism from a knot K to an alternating knot is greater than or equal to $|\nu(K)|$; that is, for all K, $\mathcal{A}_g(K) \geq |\nu(K)|$.*

Proof. Suppose that there is a genus g cobordism from K to an alternating knot J. Then $K \mathbin{\#} -J$ satisfies $g_4(K \mathbin{\#} -J) \leq g$. Thus, $|\nu(K \mathbin{\#} -J)| \leq g$. Since ν is a homomorphism and $\nu(J) = 0$, this gives $|\nu(K)| \leq g$, as desired.

□

Let $J_\eta(K)$ denote the jump in the Levine-Tristram signature function σ_K at a point η on the unit circle, [28]. For all ω that are not roots of the Alexander polynomial of K, $|\sigma_K(\omega)| \leq 2g_4(K)$. Thus, for all η, $|J_\eta(K)| \leq 4g_4(K)$. (According to [18], this jump function equals the Milnor signature [19] at η.)

To apply these jump functions to study alternating knots, we begin by considering a family of Alexander polynomials whose roots have special properties.

Lemma 4. *For $n \geq 1$, the polynomial*

$$\Delta_n(t) = t^4 + nt^3 - (2n+1)t^2 + nt + 1$$

is irreducible in $\mathbb{Q}[t]$. It has a unique root, ω_n, with positive imaginary part, and ω_n lies on the unit circle. It has two real roots, both of which are negative.

Proof. By Gauss's Lemma, we only need to show that $\Delta_n(t)$ is irreducible in $\mathbb{Z}[t]$. The only possible linear factors could be $t \pm 1$; these are ruled out by the condition that $\Delta_n(1) = 1$. The only possible quadratic factors are of the form $t^2 + at \pm 1$. The condition $\Delta_n(1) = 1$ constrains a to be in a finite set, and one can check the few possibilities.

We can rewrite $\Delta_n(t)$ as

$$\Delta_n(t) = t^2 \left((t + t^{-1})^2 + n(t + t^{-1}) - (2n + 3)\right).$$

By applying the quadratic formula, we see that roots are solutions to

$$t + \frac{1}{t} = \frac{-n \pm \sqrt{n^2 + 8n + 12}}{2}.$$

In the case that the right hand side is negative, it is clearly less than -2. The maximum negative value of $t + \frac{1}{t}$ is -2, and from this one can show that there exist two real solutions for t, both negative. In the case that the right hand side is positive, a brief calculation shows that it is less than 2. Since the minimum positive value of $t + \frac{1}{t}$ is 2, there are no corresponding real roots.

The symmetry of the polynomial implies that if α is root with positive imaginary part, then so is $\overline{\alpha^{-1}}$. There is exactly one root α with positive imaginary part, so it must be that $\alpha = \overline{\alpha^{-1}}$ and hence, α lies on the unit circle.

□

Theorem 5. *For all $n > 0$, the homomorphism J_{ω_n} vanishes on $\mathcal{C}_a$.*

Proof. We use the theorem of Murasugi [20] stating that alternating knots have Alexander polynomials of the form $\sum_{k=0}^{N} (-1)^k a_k t^k$ for some $N \geq 0$ and all $a_k > 0$. (See also, [4, 21].) In particular, for an alternating knot K, all real roots of the Alexander polynomial are positive.

Suppose that $J_{\omega_n}(K) \neq 0$. Nontrivial jumps in the signature function can occur only at roots of the Alexander polynomial, hence $\Delta_K(\omega_n) = 0$. Since $\Delta_n(t)$ is irreducible with root ω_n, it follows that $\Delta_n(t)$ divides $\Delta_K(t)$, and thus $\Delta_K(\alpha) = 0$, where α is one of the negative real roots of $\Delta_n(t)$. Thus, $\Delta_K(t)$ has a negative root, so K is not alternating. □

By a theorem of Seifert [27], there exists a knot with Alexander polynomial $\Delta_n(t)$; denote some fixed choice of such a knot by K_n. In Figure 1 we illustrate one possible choice for K_n. The reader may easily find a Seifert matrix and compute the Alexander polynomial for this knot. Notice that because one band in the Seifert surface is unknotted and untwisted, the four-genus is at most one; since the Alexander polynomial is irreducible, it is of four-genus exactly one.

We can now prove Theorem 1, which we state in more detail.

Theorem 6. *The set of knots $\{K_n\}_{n>0}$ is independent in $\mathcal{C}/\mathcal{C}_a$. For any integer $N > 0$, if K is a non-trivial knot in the span of $\{2NK_n\}$, then any cobordism from K to an alternating knot has genus at least N.*

Proof. Write $K = \sum_{i=1}^{M} a_i 2NK_i$ with some $a_i \neq 0$ and $M \geq 1$. Let n be any value of i for which $a_i \neq 0$.

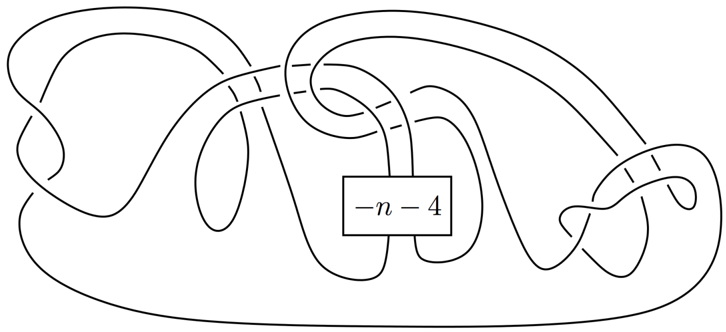

FIGURE 1. K_n

Over the real numbers, $\Delta_n(t)$ factors irreducibly as

$$\Delta_n(t) = (t-\beta_1)(t-\beta_2)(t^2 - 2\cos(\theta_n)t + 1),$$

where $\omega_n = \cos(\theta_n) + i\sin(\theta_n)$. According to Milnor [19], since $t^2 - 2\cos(\theta_n)t + 1$ is a factor of multiplicity one in $\Delta_{K_n}(t) = \Delta_n(t)$, it follows that $J_{\omega_n}(K_n) = \pm 2$. We now have $J_{\omega_n}(K) = \pm 4a_nN$. Recalling that the jump function is bounded above by $4|g_4(K)|$, Theorem 3, implies that $g_4(K) \geq |a_nN| \geq N$.

□

3. HOMOMORPHISMS ON $\mathcal{C}$ AND ALTERNATING KNOTS

Results such as those in the previous section cannot be applied to topologically slice knots; all signature invariants vanish. Thus, we are led to consider a new family of knot invariants.

Theorem 7. *Suppose that ν_1 and ν_2 are real-valued homomorphisms on $\mathcal{C}$ that satisfy $|\nu_i(K)| \leq g_4(K)$ for all K and for which $\nu_1(K) = \nu_2(K)$ for all alternating knots K. Then the homomorphism $\psi_{1,2} = \nu_1 - \nu_2$ induces a homomorphism on $\mathcal{C}/\mathcal{C}_a$, and for all knots K,*

$$\mathcal{A}_g(K) \geq \frac{1}{2}|\psi_{1,2}(K)|.$$

Proof. This is an immediate consequence of Theorem 3. □

Lemma 8. *The following homomorphisms defined on $\mathcal{C}$ bound the four-genus and are equal on alternating knots.*

- *The quotient of the classical knot signature, $\sigma(K)/2$.*
- *The negative Ozsváth-Szabó invariant, $-\tau(K)$.*
- *The quotient of the Rasmussen invariant, $s(K)/2$.*
- *The Upsilon function $\Upsilon_K(t)/t$, for each $t \in (0, 1]$.*
- *The "little upsilon function," $\upsilon(K) = \Upsilon_K(1)$.*

Proof. The fact that the first four of these invariants bound the four-genus is proved in the original references, [22, 24, 25, 26]. Futhermore, in the references [24, 25, 26] it is shown that each of $-\tau(K)$, $\Upsilon_K(t)/t$, and $s(K)/2$, agrees with $\sigma(K)/2$ for alternating knots K. The last, $\upsilon(K)$, is a specialization of $\Upsilon_K(t)/t$, used in studying alternating knots in [6]. □

4. Basic Examples

4.1. Examples based on σ, τ and s.

Example 9. One can compute that $\sigma(T(3,7))/2 = -4$ and $-\tau(T(3,7)) = -6$; hence $|\sigma(T(3,7))/2 - (-\tau(T(3,7)))| = 2$. Thus, by Theorem 7 any cobordism from $T(3,7)$ to an alternating knot must have genus at least one.

Example 10. Similarly, according to Hedden and Ording [11], the values of $-\tau$ and $s/2$ differ by one on the twice-twisted positive Whitehead double of the trefoil knot, $\mathrm{Wh}^+(T(2,3),2)$, and thus this knot is not concordant to an alternating knot. Here $\mathrm{Wh}^+(\,\cdot\,,2)$ denotes the positive clasped, twice twisted, Whitehead double and $C_{n,2n-1}(\,\cdot\,)$ denotes the $(n,2n-1)$–cable.

Notice that $-\tau$ and $s/2$ agree on $T(3,7)$ and so we find that $T(3,7)$ and $\mathrm{Wh}^+(T(2,3),2)$ generate a rank two free subgroup of $\mathcal{C}/\mathcal{C}_a$: the pair of homomorphisms $\sigma(\cdot)/2 - \tau(\cdot)$ and $\tau(\cdot) + s(\cdot)/2$ define an injection from the span of these two knots in $\mathcal{C}/\mathcal{C}_a$ to $\mathbb{Z} \oplus \mathbb{Z}$.

Example 11. One can use τ and σ to give examples of topologically slice knots that are nontrivial in $\mathcal{C}/\mathcal{C}_a$. According to [15], the untwisted Whitehead double of the trefoil knot, $\mathrm{Wh}^+(T(2,3),0)$, has $\tau = 1$ and $\sigma = 0$. Thus, it is not concordant to an alternating knot.

4.2. Examples based on the Upsilion invariant.

For any $s, t \in (0,1]$, we let $\psi_{s,t}(K) = \Upsilon_K(s)/s - \Upsilon_K(t)/t$. There is an immediate consequence of Theorem 7.

Theorem 12. *For all knots K, $\mathcal{A}_g(K) \geq \frac{1}{2}|\psi_{t,s}(K)|$.*

Example 13. Consider the difference $K = T(3,7) \# - T(4,5)$. Figure 2 illustrates the function $\Upsilon_K(t)/t$. Now let $\psi_{t,s}(K) = \Upsilon_K(s)/s - \Upsilon_K(t)/t$. As the graph indicates, $|\psi_{0,2/3}(K)| = 1$. Thus, any cobordism from K to an alternating knot must have genus at least 1. This could not have been determined using $\tau(K) = 0$ or $\upsilon(K) = 0$. Similarly, separate computations show that both the Rasmussen invariant and the signature vanish. Thus, to detect this knot, we must use $\Upsilon_K(t)$ for some $t \in (0,1)$.

4.3. An infinite independent family in $\mathcal{C}/\mathcal{C}_a$.

By using different values of s and t, we prove the following result.

Theorem 14. (1) *The set of knots $\{T(p,p+1)\}$, $p \geq 3$, is linearly independent in $\mathcal{C}/\mathcal{C}_a$.*
(2) *For every integer $N > 0$, there exists a sequence of positive integers $\{N_p\}$ such that the set $S = \{N_p T(p,p+1)\}$ has the property that every non-trivial knot in span(S) satisfies $\mathcal{A}_g(J) \geq N$.*

Proof. To simplify notation, let $K_p = T(p,p+1)$. Using the methods of [25], we have that the first singularity of $\Upsilon_{K_p}(t)$ is at $t = 2/p$ and the second singularity is at $4/p$.

For each p, choose an ϵ_p with $2/p < 2/p + \epsilon_p < 2/(p-1)$. Consider the functions $\phi_{2/p,2/p+\epsilon_p}$, which we abbreviate ψ_p for the moment. Then $\psi_p(K_p) \neq 0$ and $\psi_p(K_n) = 0$

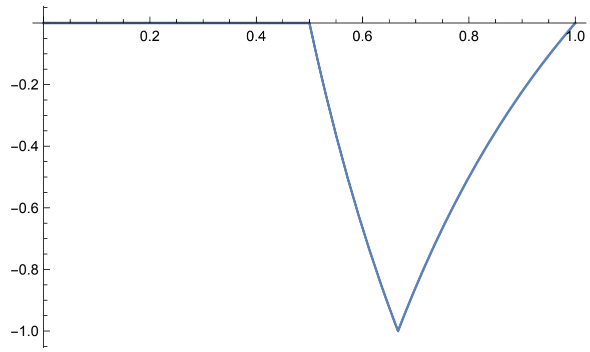

FIGURE 2. $\Upsilon_{T(3,7)-T(4,5)}(t)/t$

for all $n < p$. (We use here the fact that $\Upsilon_K(0) = 0$ for all knots K, so $\Upsilon_{K_p}(t) = c_p t$ on the interval $t \in [0, 2/p]$ for some c_p.)

Suppose that some finite linear combination is trivial: $\sum a_n K_n = 0 \in \mathcal{C}_a$. Let p be the largest value of n for which $a_n \neq 0$. Applying ψ_p to the sum yields a nonzero multiple of a_n. But ψ_p would vanish if the sum were in $\mathcal{C}_a$, implying that $a_p = 0$, a contradiction. This completes the proof of linear independence.

For statement (2), for each p select any integer N_p so that $\psi_p(N_p T(p, p+1)) \geq 2N$. Following the same argument as in (1), let $J_p = N_p T(p, p+1)$. If $J = \sum a_n J_n = 0 \in \mathcal{C}_a$, let p be the largest n for which $a_n \neq 0$. Then $|\psi_p(J)| = |\psi_p(J_p)| \geq 2N$.

□

5. PROOF OF THEOREM 2; AN INFINITE FAMILY OF TOPOLOGICALLY SLICE KNOTS.

We use the examples developed in [25]. For $n \geq 2$, let

$$K_n = \mathrm{C}_{n,2n-1}(\mathrm{Wh}^+(T(2,3),0))\#(-T(n,2n-1)).$$

Here $\mathrm{C}_{n,2n-1}(\ \cdot\ )$ denotes the $(n, 2n-1)$–cable. This knot is topologically slice; to see this, observe that the untwisted double is topologically slice, so its $(n, 2n-1)$–cable is topologically concordant to $T(n, 2n-1)$. The proof of [25, Theorem 1.20] includes a computation of certain values of $\Upsilon_{K_n}(t)$. In summary:

Theorem 15. *For each $n > 0$ there is an $\epsilon_n > 0$ such that*

$$\begin{cases} \Upsilon_{K_n}(t) = 0 & \text{if } t \leq \frac{2}{2n-1} \\ \Upsilon_{K_n}(t) > 0 & \text{if } \frac{2}{2n-1} < t < \frac{2}{2n-1} + \epsilon_n. \end{cases}$$

To simplify notation, we let $a_n = \frac{2}{2n-1}$ and $b_n = a_n + \epsilon_n$. Each ϵ_n can be chosen so that $b_{n+1} < a_n$.

We now let $\psi_n = \phi_{a_n, b_n}$. As an immediate corollary to Theorem 15 we have the following.

Corollary 16. *For all $n > 0$, $\psi_n(K_n) > 0$ and $\psi_n(K_m) = 0$ for $m < n$.*

5.1. **Proof of Theorem 2.** The proof of Theorem 2 is identical to that given above for Theorem 14, with the family of knots K_n replacing the torus knots $T(p, p+1)$, and using Corollary 16 instead of the simpler result concerning the values of $\Upsilon_{T(p,p+1)}(t)$. The subgroup $\mathcal{H}_N$ is now generated by the set of knots $\{N_n K_n\}$ for appropriately chosen N_n.

6. Singular concordances

In this section we will consider the count of double points in singular concordances between knots. In the case that one of the knots is the unknot, this is a well studied invariant; references include [3, 13, 23].

It is known that each of the knot invariants $\sigma(K), -\tau(K), s(K)/2$, and $\Upsilon_K(t)/t$, remains unchanged or increases by one if a positive crossing is changed to be a negative crossing. This fact is implied by the following stronger theorem. The proof is related to the proof of the crossing change bounds for τ and $\Upsilon_K(t)/t$ given in [15, 16]. We give the argument for more general invariants.

Lemma 17. *Let ν be a homomorphism from $\mathcal{C}$ to $\mathbb{R}$ satisfying*

(1) *For all K, $|\nu(K)| \leq g_4(K)$, and*
(2) *$\nu(J) = -1$ for some knot J with the property that changing a single crossing in J from positive to negative yields a slice knot.*

Suppose there is a singular concordance from K_1 and K_2 with precisely $s = s_+ + s_-$ double points, where s_+ is the number of positive double points. Then

$$-s_+ \leq \nu(K_1) - \nu(K_2) \leq s_-.$$

Proof. We show that for a knot K bounding a singular disk with s_+ positive double points and s_- negative double points,

$$-s_+ \leq \nu(K) \leq s_-.$$

This can be applied to $K = K_1 \# - K_2$ to complete the proof of the theorem.

We discuss the case of $s_- \leq s_+$. The singular disk can be converted into an embedded surface by tubing together s_- pairs of positive and negative crossing points, and then resolving the remaining $s_+ - s_-$ double points. This yields a surface bounded by K of genus s_+. Thus, $g_4(K) \leq s_+$, so we also have $|\nu(K)| \leq s_+$, and in particular, $-s_+ \leq \nu(K)$.

Notice that $K \# -(s_+ - s_-)J$ bounds a singular disk with s_+ negative double points as well as s_+ positive double points. By tubing the double points together, we have $g_4(K \# -(s_+ - s_-)J) \leq s_+$. It follows that

$$|\nu(K \# -(s_+ - s_-)J)| \leq s_+.$$

Using the fact that $\nu(J) = -1$, this gives

$$|\nu(K) + s_+ - s_-| \leq s_+,$$

and in particular, $\nu(K) \leq s_-$. This completes the proof in the first case, $s_- \leq s_+$. The second case can be proved similarly, or one can apply the first case to the knot $-K$.

□

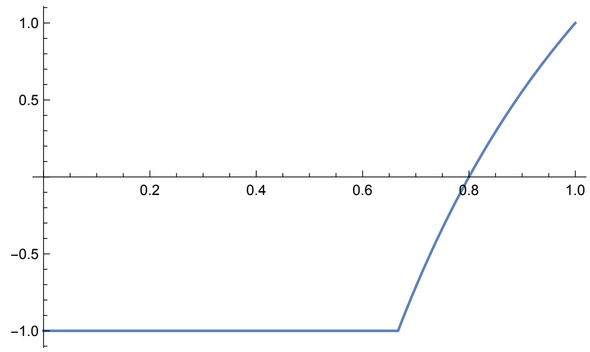

FIGURE 3. $\Upsilon_{T(3,7)\#-T(2,11)}(t)/t$

Note. For each of the knot invariants we are considering, the knot J can be taken to be $T(2,3)$.

The following bound on $\mathcal{A}_s(K)$ is similar to the one given on $\mathcal{A}_g$ in Theorem 12; note, however, that the bound has been doubled.

Theorem 18. *Let ν_1 and ν_2 be any two homomorphisms on $\mathcal{C}$ that satisfy the two conditions of Theorem 17 and which agree on alternating knots. Then the homomorphism $\psi_{1,2} = \nu_1 - \nu_2$ induces a homomorphism on $\mathcal{C}/\mathcal{C}_a$, and for all knots K,*

$$\mathcal{A}_s(K) \geq |\psi_{1,2}(K)|.$$

Proof. Suppose that there is a singular concordance from K to an alternating knot K' with s_+ positive double points and s_- negative double points. Let $s = s_+ + s_-$. By Theorem 17,

$$-s_+ \leq \nu_1(K) - \nu_1(K') \leq s_-$$

and

$$-s_- \leq \nu_2(K') - \nu_2(K) \leq s_+.$$

Adding these and using the fact that $\nu_1(K') = \nu_2(K')$ shows

$$-s \leq \nu_1(K) - \nu_2(K) \leq s,$$

which can be written in terms of $\psi_{1,2}$ as $|\psi_{1,2}(K)| \leq s$. This hold for all possible s (including its minimum value) so $|\psi_{1,2}(K)| \leq \mathcal{A}_s(K)$ as desired. □

Example 19. Consider $J = T(3,7)\# - T(2,11)$. An illustration of $\Upsilon_J(t)$ is given in Figure 3. For this knot, we compute the difference $|\upsilon(J) - \tau(J)| = 2$. Thus, any singular concordance from J to an alternating knot must have at least two singular points. On the other hand, a result of Feller [5] implies that this knot has four-genus one, and hence there is a genus one cobordism from J to an alternating knot. Two crossing changes convert $T(3,7)$ into an alternating knot (see, for instance [12]), so $\mathcal{A}_s(J) \leq 2$.

In summary, for the knot $J = T(2,11) - T(3,7)$, $\mathcal{A}_g(J) = 1$ and $\mathcal{A}_s(J) = 2$. That is, both bounds from Theorem 2 are realized.

There is an immediate theorem, parallel to Theorem 2.

Theorem 20. *For every positive integer N, there is an infinitely generated free subgroup $\mathcal{H}_N \subset \mathcal{C}/\mathcal{C}_a$ with the following properties.*

(1) *If K represents a nontrivial class in $\mathcal{H}_N$, then any generic singular concordance from K to an alternating knot has at least N double points.*
(2) *Every class in $\mathcal{H}_N$ is represented by a topologically slice knot; in particular, the quotient $\mathcal{C}_{ts}/(\mathcal{C}_{ts} \cap \mathcal{C}_a)$ is infinitely generated, where $\mathcal{C}_{ts}$ is the concordance group of topologically slice knots.*

7. Questions

It has been shown that the quotient $\mathcal{C}_{ts}/(\mathcal{C}_{ts} \cap \mathcal{C}_a)$ is infinitely generated; informally, $\mathcal{C}_{ts} \cap \mathcal{C}_a$ is small compared to $\mathcal{C}_{ts}$. It is possible that $\mathcal{C}_{ts} \cap \mathcal{C}_a = 0$. Thus, one can ask the following question.

Question 21. *Is there an alternating knot that is topologically slice but not smoothly slice?*

Initially, the only known examples of topologically slice knots that are not smoothly slice arose directly from Freedman's theorem [8] that knots K with $\Delta_K(t) = 1$ are topologically slice. Non-trivial knots with trivial Alexander polynomial are not alternating, so this result is probably not of immediate use in resolving Question 21. Freedman's theorem was generalized in [7] to a class of knots with nontrivial alternating Alexander polynomials, offering a possible source of examples, and in [9, 10] infinite families of slice knots which are not even concordant to polynomial one knots were constructed. Unfortunately, it is not clear how to use those results to build alternating examples.

In a different direction, Litherland [14] proved that the set of positive torus knots is linearly independent in $\mathcal{C}$. Clearly, they become dependent in $\mathcal{C}/\mathcal{C}_a$, since the torus knots $T(2, 2n+1)$ are alternating.

Question 22. *Is the set of torus knots $\{T(p,q)\}$ with $3 \le p < q$ independent in $\mathcal{C}/\mathcal{C}_a$?*

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