Exercise Sheet 2

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Exercices labeled with a star $*$ are going to be graded. Please hand in solutions to Felix Eberhart not later than Monday 31 May at noon.

Exercice 1*. Let $\pi\colon P \to M$ be a principal fibre bundle with a connection A over a connected base M. Parallel transport along a closed loop $\gamma\colon [0,1] \to M$ with $\gamma(0) = \gamma(1) = x$, starting at $u \in \pi^{-1}(x)$, defines an element $\in G$ by the formula

$$Par^A_\gamma(u) =: u \cdot hol^A_u(\gamma)$$

This is called the *holonomy* of γ centered at u with respect to the connection A.

(a) Show that $hol^A_u(\gamma * \delta) = hol^A_u(\gamma)\, hol^A_u(\delta)$, where $*$ denotes the composition of paths. (We use the maybe unusual convention that the path $\gamma * \delta$ denotes the path which first follows δ, then γ.)

(b) Show that $hol_{ug} = g^{-1} hol_u\, g$.

(c) Suppose $\mu\colon [0,1] \to M$ is some path on M. Then we have

$$hol^A_{Par^A_\mu(u)}(\mu * \gamma * \mu^-) = hol^A_u(\gamma).$$

Here, μ^- denotes the inverse path.

Taking holonomies along closed paths based at u defines a subgroup $Hol_u(A)$ of G, called the holonomy group of A centered at u, and taking holonomies of loops homotopic to the constant loop defines a subgroup $Hol^0_u(A)$ which is called the reduced holonomy group.

(d) Show that $Hol^0_u(A)$ is a normal subgroup of $Hol_u(A)$.

(e) Show the *Holonomy reduction theorem*: Let $P^A(u)$ denote the set of points that can be reached by parallel transport along (not only closed) paths starting from u. Then this defines a reduction of the structure group from G to $Hol_u(A)$ of the bundle $\pi\colon P \to M$.

(f) Show the *Theorem of Ambrose and Singer*: Then the Lie algebra $\mathfrak{hol}_u(A)$ of the holonomy group $Hol_u(A)$ is given as follows:

$$\mathfrak{hol}_u(A) = \{\Omega^A_q(\xi,\zeta)\,|\,q \in P^A(u), \xi,\zeta \in TP\},$$

where Ω^A denotes the curvature of A.

(g) Show that if A is a flat connection, then the holonomy descends to a group homomorphism

$$hol\colon \pi_1(M,x) \to G$$
$$[\gamma] \mapsto hol^A_u(\gamma).$$

Exercice 2*. Let E be a $U(2)$-vector bundle, and P_E its associated unitary frame bundle, a $U(2)$-principal fibre bundle. We denote by $\mathfrak{su}(E)$ the subbundle of the endomorphism-bundle $\mathrm{End}(E) \cong E \otimes E^*$ consisting of trace-free and skey-adjoint endomorphisms. The aim of this exercise is to show that we have

$$p_1(\mathfrak{su}(E)) = -4c_2(E) + c_1(E)^2.$$

(a) Show that $\mathfrak{su}(E) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}_0(E)$, where $\mathrm{End}_0(E)$ denotes the trace-free endomorphisms of E. Show that there is a canonical isomorphism $\mathrm{End}_0(E) \oplus \mathbb{C} \cong \mathrm{End}(E)$.

(b) By definition of the first Pontryagin class we have

$$p_1(\mathfrak{su}(E)) = -c_2(\mathfrak{su}(E) \otimes_{\mathbb{R}} \mathbb{C}).$$

Use (a) and the Chern-character (which is multiplicative with respect to tensor product) to derive the formula above expressing $p_1(\mathfrak{su}(E))$ by the Chern classes of E.

(c) Derive this same formula using Chern-Weil theory: Express $\mathfrak{su}(E)$ as an associated bundle of P_E, and push forward a connection from P_E to $\mathfrak{su}(E)$, and conclude.

Exercice 3. Show that $U(n)$-vector bundles on 4-manifolds are classified, up to isomorphism, by their first and second Chern class. Show that this does not hold in higher dimensions. (Hint for the latter: Use that $BSU(2) \cong \mathbb{HP}^1 \cong S^4$, and that G-vector bundles on a topological space X are classified by $[X, BG]$, homotopy classes of maps $X \to BG$. Use a model from the literature for $BSU(2)$, and use cellular approximation to reduce it to $[S^5, S^4] \cong \mathbb{Z}/2$.).