# HOMOLOGY LENS SPACES AND $\mathrm{SL}(2, \mathbb{C})$

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Abstract. We prove that if $Y$ is a closed, oriented 3-manifold with first homology $H_1(Y;\mathbb{Z})$ of order less than 5, then there is an irreducible representation $\pi_1(Y) \to \mathrm{SL}(2,\mathbb{C})$ unless $Y$ is homeomorphic to $S^3$, a lens space, or $\mathbb{RP}^3\#\mathbb{RP}^3$. By previous work it suffices to consider the case $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$, which we accomplish using holonomy perturbation techniques in instanton Floer homology.

## 1. Introduction

If we fix an integer $n \geq 4$, then a classical construction demonstrates that any finitely presented group $G$ can be realized as the fundamental group of some closed $n$-manifold. The analogous statement is false in three dimensions, leading one to ask which $G$ can actually be realized as 3-manifold groups. Techniques from both gauge theory and hyperbolic geometry suggest that representation varieties could be a good source of restrictions on $G$, and indeed the third author used these to prove the following:

**Theorem 1.1** ([Zen18])**.** *Let $Y$ be an integer homology 3-sphere. If $Y$ is not homeomorphic to $S^3$, then there is an irreducible representation $\pi_1(Y) \to \mathrm{SL}(2,\mathbb{C})$.*

In recent work we generalized this to the cases where $H_1(Y)$ is 2-torsion or 3-torsion.

**Theorem 1.2** ([GSZ23, Theorem 1.2])**.** *Let $Y$ be a closed, oriented, connected 3-manifold with $H_1(Y;\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus r}$. If $Y$ is not homeomorphic to $\#^r\mathbb{RP}^3$, then there is an irreducible representation $\pi_1(Y) \to \mathrm{SL}(2,\mathbb{C})$.*

**Theorem 1.3** ([GSZ23, Theorem 1.3])**.** *Let $Y$ be a closed, oriented, connected 3-manifold with $H_1(Y;\mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^{\oplus r}$ for some $r \geq 1$. If $Y$ is not homeomorphic to $\pm L(3,1)$, then there is an irreducible representation $\pi_1(Y) \to \mathrm{SL}(2,\mathbb{C})$.*

The main goal of this paper is to generalize the techniques of [GSZ23] and add the following additional case.

**Theorem 1.4.** *Let $Y$ be a closed, orientable 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$, and suppose that $Y$ is not homeomorphic to a lens space. Then there is an irreducible representation $\pi_1(Y) \to \mathrm{SL}(2,\mathbb{C})$.*

**Corollary 1.5.** *Let $Y$ be a closed, orientable 3-manifold with $|H_1(Y;\mathbb{Z})| < 5$. Then there is an irreducible representation $\pi_1(Y) \to \mathrm{SL}(2,\mathbb{C})$ unless $Y$ is $S^3$, a lens space, or $\mathbb{RP}^3\#\mathbb{RP}^3$.*

*Proof.* When $H_1(Y) = 0$, this is the main result of [Zen18]. The cases where $H_1(Y)$ is $\mathbb{Z}/2\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ are part of Theorem 1.2, while Theorem 1.3 handles the case $H_1(Y) \cong \mathbb{Z}/3\mathbb{Z}$. This leaves only $\mathbb{Z}/4\mathbb{Z}$, which is Theorem 1.4. □

1.1. **Strategy.** The proof of Theorem 1.4 follows the same broad outline as the results of [Zen18, GSZ23]. We say that $Y$ is $\mathrm{SL}(2,\mathbb{C})$-*reducible* if every representation $\pi_1(Y) \to \mathrm{SL}(2,\mathbb{C})$ has reducible image, and then our goal is to classify the $\mathrm{SL}(2,\mathbb{C})$-reducible 3-manifolds $Y$ with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$. By appealing to [Zen18] we see that any such $Y$ must be irreducible, and then the cases where $Y$ is hyperbolic or Seifert fibered are readily dispatched by appeals to [CS83] and [SZ22a] respectively. Thus by geometrization it suffices to assume that $Y$ has an incompressible torus.

*Simplification of toroidal counterexamples.* We now write our $\mathrm{SL}(2,\mathbb{C})$-reducible, toroidal manifold as $Y = M_1 \cup_{T^2} M_2$, where $M_1$ and $M_2$ are compact, irreducible 3-manifolds with incompressible torus boundaries. If there is a degree-1 map of the form $f : M_1 \to M_1'$ that restricts to a homeomorphism $\partial M_1 \xrightarrow{\cong} \partial M_1'$, then we can extend this to a degree-1 map of the form

$$Y = M_1 \cup_{T^2} M_2 \to M_1' \cup_{T^2} M_2 = Y',$$

where the extension to $M_2$ is by the identity. (To be precise, we can use the gluing map $\partial M_1 \cong \partial M_2$ coming from $Y$ and pick an identification $\partial M_1' \cong \partial M_2$ so that the homeomorphism $f|_{\partial M_1} : \partial M_1 \to \partial M_1'$, read through these identifications, becomes the identity on $\partial M_2$, and this obviously extends by the identity.) As a degree-1 map, this induces a surjection $\pi_1(Y) \to \pi_1(Y')$, implying that $Y'$ must also be $\mathrm{SL}(2,\mathbb{C})$-reducible. The same holds for degree-1 maps $M_2 \to M_2'$, so we may simplify $Y$ by applying as many such maps as we like.

In favorable situations there is a natural source of degree-1 maps. Each of the $M_j$ comes equipped with a *rational longitude*, uniquely defined up to sign, which is a primitive class

$$\lambda_j \in H_1(\partial M_j;\mathbb{Z})$$

whose image in rational homology generates the kernel of the rank-1 map $H_1(\partial M_j;\mathbb{Q}) \to H_1(M_j;\mathbb{Q})$. If we assume that $H_1(Y) \cong \mathbb{Z}/4\mathbb{Z}$, then it turns out that either

(1) $\lambda_j$ is nullhomologous, and there is a classical degree-1 "pinching" map $M_j \to S^1 \times D^2$ (see [GSZ23, Proposition 4.2] for details);
(2) or $\lambda_j$ has order 2 in $H_1(M;\mathbb{Z})$, and then we constructed an analogous pinching map from $M_j$ onto the twisted $I$-bundle over the Klein bottle [GSZ23, Proposition 4.5].

We manage to eliminate the second possibility, by using an explicit understanding of the $\mathrm{SU}(2)$ character variety of the twisted $I$-bundle over the Klein bottle to construct non-abelian representations $\pi_1(Y') \to \mathrm{SU}(2)$. Thus for the main theorem it ultimately suffices to assume that both $M_1$ and $M_2$ are homology solid tori, and that the Dehn fillings $M_1(\lambda_2)$ and $M_2(\lambda_1)$ are both lens spaces of order 4.

*Curves of characters in the pillowcase.* Once we have simplified each $M_j$ as above, we can use techniques from gauge theory to construct a non-abelian representation $\pi_1(Y) \to \mathrm{SU}(2)$. It suffices to find a pair of representations $\rho_j : \pi_1(M_j) \to \mathrm{SU}(2)$ that agree when restricted to $\pi_1(T^2)$, so we study the images of the $\mathrm{SU}(2)$ character varieties $X(M_j)$ inside the *pillowcase* $X(T^2)$, the $\mathrm{SU}(2)$ character variety of $T^2$. If these images intersect at a point where at least one of the corresponding $\rho_j$ is non-abelian, then these $\rho_j$ will glue to give the desired non-abelian $\rho : \pi_1(Y) \to \mathrm{SU}(2)$.

Having arranged that each $M_j$ is a homology solid torus with a lens space filling, it follows that the pillowcase image of $M_j$ must

(1) avoid certain line segments in the pillowcase, where non-abelian representations $\pi_1(M_j) \to \mathrm{SU}(2)$ would otherwise descend to non-abelian representations of the corresponding lens space;
(2) and contain an essential closed curve in the twice-punctured pillowcase (which is homeomorphic to an open cylinder).

Claim (2) is Proposition 5.5; to prove it, we argue that the longitudinal filling $M_j(\lambda_j)$ must be irreducible and thus have non-vanishing instanton homology, and then the existence of the essential curve follows from the techniques of [Zen18] and [LPCZ23].

We now have a pair of simple closed curves $\gamma_1$ and $\gamma_2$ in the pillowcase, corresponding respectively to SU(2) representations of $\pi_1(M_1)$ and of $\pi_1(M_2)$, and we wish to show that they intersect. In fact, there is a slight caveat: there is exactly one point $p \in X(T^2)$, denoted $(\frac{\pi}{2}, 0)$ in our preferred coordinates, that might correspond to abelian representations of both of the $\pi_1(M_j)$. But if the $\gamma_i$ meet at $p$ then they must do so transversely, and then they must also meet somewhere else, because simple closed curves on $X(T^2) \cong S^2$ cannot have a single transverse point of intersection. So it suffices to show that $\gamma_1$ intersects $\gamma_2$.

From here the most novel part of the argument, in comparison with our work [GSZ23], is the case where exactly one of the $\gamma_j$ (say, $\gamma_2$) passes through the distinguished point $p$. Writing each $M_j$ as the complement $Y_j \setminus N(K_j)$ of a knot in a homology sphere, we used a non-vanishing theorem of the form

$$I_*^{w_j}\big((Y_j)_0(K_j)\big) \neq 0,$$

in conjunction with the holonomy perturbation techniques of [Zen18], to deduce the existence of the curves $\gamma_j$. Here, by contrast, we apply these techniques to a lower bound

$$\dim \mathit{KHI}(Y_j, K_j) > 1$$

on the *instanton knot homology* of [KM10]. Specifically, we use $\gamma_2$, together with the line avoided by $X(M_1)$ as in claim (1) above, to construct a simple closed curve $c$ in the pillowcase with the properties that

(1) if the irreducible characters of $M_1$ avoid the curve $\gamma_2$ in the pillowcase, then they also avoid $c$;
(2) and $c$ divides the pillowcase into two regions of equal area.

Following [SZ22b, §4], the holonomy perturbation argument says that the representations $\pi_1(M_1) \to \mathrm{SU}(2)$ along some $C^1$-small approximation of $c$ generate a chain complex for $\mathit{KHI}(Y_1, K_1)$, and only one of these generators corresponds to an abelian representation, so there must be a non-abelian representation $\rho_1 : \pi_1(M_1) \to \mathrm{SU}(2)$ whose image meets $\gamma_2$ after all. This gives rise to the desired representation $\rho$ of $\pi_1(Y)$, completing the proof.

1.2. **Organization.** We begin with a brief review of the pillowcase in §2. In §3 we reduce Theorem 1.4 to the case of toroidal manifolds $Y$ where one side of the incompressible torus has a lens space filling, and either the other side is either a homology solid torus with a lens space filling or its rational longitude has reasonably small order. Then in §4 we rule out the second possibility by reducing it to the case where that side is the twisted $I$-bundle over the Klein bottle. Finally, most of the paper is concentrated in §5, where we use instanton gauge theory to associate curves of characters in the pillowcase to each piece of $Y \setminus N(T^2)$ and thus prove that the images of their character varieties must intersect.

## 2. The pillowcase

In this section we briefly review some facts about the pillowcase orbifold, as discussed for example in [LPCZ23, §3.1] or [GSZ23, §3.1].

Given a manifold $M$, we can define the SU(2) representation and character varieties of $M$ by

$$\begin{aligned} R(M) &= \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2)), \\ X(M) &= R(M)/\mathrm{SO}(3), \end{aligned}$$

where $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$ acts on $R(M)$ by conjugation. We write $R^{\mathrm{irr}}(M)$ or $X^{\mathrm{irr}}(M)$ for the subsets of each variety consisting of irreducible representations. If $M$ is the exterior of a knot $K \subset Y$ then we will also write $R(Y, K)$ to mean $R(M)$ and so on. We will say that $M$ is SU(2)-*abelian* if $R(M)$ consists entirely of representations with abelian image.

In the case $M = T^2$, we have $\pi_1(T^2) \cong \mathbb{Z}^2$, say with generators $\mu$ and $\lambda$. A representation $\rho : \pi_1(T^2) \to \mathrm{SU}(2)$ is then determined by a pair of commuting matrices $\rho(\mu)$ and $\rho(\lambda)$, and since these are in SU(2) they can be simultaneously diagonalized, so that up to conjugacy we have

$$\rho(\mu) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}, \qquad \rho(\lambda) = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} \tag{2.1}$$

for some $\alpha, \beta \in \mathbb{R}/2\pi\mathbb{Z}$. This uniquely determines the conjugacy class of $\rho$ up to replacing $(\alpha, \beta)$ with $(-\alpha, -\beta)$, so we have

$$X(T^2) \cong \frac{(\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})}{(\alpha, \beta) \sim (2\pi - \alpha, 2\pi - \beta)}.$$

This quotient is homeomorphic to a sphere, but has four orbifold points of order 2 (where $\alpha, \beta \in \pi\mathbb{Z}$), so it is known as the *pillowcase orbifold*. In practice we will use a fundamental domain for this and write

$$X(T^2) \cong \frac{[0, \pi]_\alpha \times [0, 2\pi]_\beta}{\left\{ \begin{array}{c} (0, \beta) \sim (0, 2\pi - \beta) \\ (\pi, \beta) \sim (\pi, 2\pi - \beta) \\ (\alpha, 0) \sim (\alpha, 2\pi) \end{array} \right\}}.$$

Now if $M$ is a compact 3-manifold with boundary $T^2$, the inclusion

$$i : T^2 \cong \partial M \hookrightarrow M$$

gives a homomorphism $i_* : \pi_1(T^2) \to \pi_1(M)$ and hence a pullback map

$$i^* : X(M) \to X(T^2).$$

We refer to the image $i^*X(M)$ or $i^*X^{\mathrm{irr}}(M)$ as the *pillowcase image* of $M$.

**Example 2.1.** In the case where $M \cong Y \setminus N(K)$ is the complement of a nullhomologous knot $K \subset Y$, we use the meridian $\mu$ and longitude $\lambda$ of $K$ as our preferred generators of $\pi_1(T^2)$, giving rise to coordinates $(\alpha, \beta)$ on the pillowcase as in (2.1). Any abelian representation $\rho : \pi_1(M) \to \mathrm{SU}(2)$ will factor through $H_1(M)$, where $[\lambda] = 0$, and thus satisfy $\rho(\lambda) = 1$. Thus in $i^*X(Y, K)$ the abelian representations will all lie on the line $\beta \equiv 0 \pmod{2\pi}$, and conversely any representation off that line must be non-abelian.

**Lemma 2.2** ([LPCZ23, Lemma 3.1]). *Let $Y$ be an* SU(2)*-abelian integer homology sphere, and let $K \subset Y$ be a knot. Then there is some positive $\delta = \delta(K) > 0$ such that the pillowcase image*

$$i^* X^{irr}(Y, K) \subset X(T^2)$$

*is contained in the region $\delta \le \alpha \le \pi - \delta$.*

In the situation of Lemma 2.2, we can lift $i^*X(Y, K)$ to the *cut-open pillowcase*

$$\mathcal{C} = [0, \pi]_\alpha \times (\mathbb{R}/2\pi\mathbb{Z}),$$

because Example 2.1 and Lemma 2.2 tell us that the image $i^*X(Y, K)$ only meets the lines $\alpha = 0$ and $\alpha = \pi$ in the pillowcase at the points $(\alpha, \beta) = (0, 0)$ and $(\pi, 0)$, both of which have a unique preimage in $\mathcal{C}$. Lidman, Pinzón-Caicedo, and Zentner [LPCZ23] proved the following, generalizing a theorem of Zentner [Zen18, Theorem 7.1] in the case $Y = S^3$.

**Theorem 2.3** ([LPCZ23]). *Let $Y$ be an* SU(2)*-abelian integer homology sphere, and let $K \subset Y$ be a knot with irreducible, boundary-incompressible exterior. Then the image*

$$i^*X(Y, K) \subset \mathcal{C}$$

*in the cut-open pillowcase contains a topologically embedded curve in the interior of $\mathcal{C}$ that is homologically nontrivial in $H_1(\mathcal{C}; \mathbb{Z}) \cong \mathbb{Z}$.*

*Proof.* Since $Y$ is an SU(2)-abelian homology sphere, its instanton homology $I_*(Y)$ is zero. Then by [LPCZ23, Theorem 1.3], the fact that $Y \setminus N(K)$ is irreducible and boundary-incompressible implies the non-vanishing of $I^w_*(Y_0(K))$, where $w \in H^2(Y_0(K); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is nonzero. Now we can apply the "pillowcase alternative" of [LPCZ23, Theorem 3.5] to find the desired curve $\gamma$ in $i^*X(Y, K)$.

The only claim that needs further justification is that $\gamma$ avoids the boundary curves $\{\alpha = 0\}$ and $\{\alpha = \pi\}$. We take the constant $\delta > 0$ provided by Lemma 2.2 and note that by Example 2.1, any point $(\alpha, \beta) \in \gamma$ with either $0 \le \alpha < \delta$ or $\pi - \delta < \alpha \le \pi$ must satisfy $\beta \equiv 0 \pmod{\pi}$. If the intersection

$$I_1 = \gamma \cap \big([0, \delta/2] \times (\mathbb{R}/2\pi\mathbb{Z})\big) \subset \mathcal{C}$$

is nonempty and we let

$$\alpha_1 = \inf\{\alpha \in [0, \delta/2] \mid (\alpha, 0) \in \gamma\},$$

then $I_1$ must therefore be precisely the line segment $[\alpha_1, \delta/2] \times \{0\} \subset \mathcal{C}$: we have $(\alpha_1, 0) \in \gamma$ since $\gamma$ is closed, and if any point of $(\alpha_1, \delta/2] \times \{0\}$ were missing from $\gamma$ then $\gamma$ would be disconnected. By the same argument the intersection

$$I_2 = \gamma \cap \big([\pi - \delta/2, \pi] \times (\mathbb{R}/2\pi\mathbb{Z})\big) \subset \mathcal{C}$$

must either be empty or have the form $[\pi - \delta/2, \alpha_2] \times \{0\}$. We let $f_s : [0, \pi] \to [0, \pi]$ be a deformation retraction of $[0, \pi]$ onto $[\delta/2, \pi - \delta/2]$, and then

$$\mathcal{C} = [0, \pi] \times (\mathbb{R}/2\pi\mathbb{Z}) \xrightarrow{f_s \times \mathrm{id}} [0, \pi] \times (\mathbb{R}/2\pi\mathbb{Z}) = \mathcal{C}$$

is a deformation retraction of $\mathcal{C}$ that restricts to a deformation retraction of $\gamma$. We replace $\gamma$ with its image $f_1(\gamma) \subset i^*X(Y, K)$, which is the desired curve because it no longer contains any point $(\alpha, \beta)$ with $\alpha < \delta/2$ or $\alpha > \pi - \delta/2$. □

## 3. Decompositions of toroidal manifolds

In this section, which parallels [GSZ23, §7], we show that a toroidal manifold $Y$ with homology $H_1(Y) \cong \mathbb{Z}/p^e\mathbb{Z}$ of prime power order must be a union of two relatively simple pieces. Assuming that such a manifold is $\mathrm{SL}(2,\mathbb{C})$-reducible, Proposition 3.5 will provide a minimal example with respect to the partial ordering given by degree-1 maps. If we further assume that $H_1(Y) \cong \mathbb{Z}/4\mathbb{Z}$, then we will show in Proposition 4.4 that both pieces must have nullhomologous rational longitudes, so that we can focus on this case afterward.

**Lemma 3.1.** *Let $Y$ be a closed, connected, orientable, toroidal 3-manifold with $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/p^e\mathbb{Z}$ for some prime $p$ and integer $e \geq 0$. Then we can write*

$$Y = M_1 \cup_{T^2} M_2,$$

*where $M_1$ and $M_2$ are compact manifolds with incompressible torus boundary satisfying*

$$H_1(M_1;\mathbb{Z}) \cong \mathbb{Z},$$
$$H_1(M_2;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/p^f\mathbb{Z}$$

*for some integer $f$ with $0 \leq f \leq e$.*

*Proof.* A torus in a rational homology sphere must be separating, so we can write $Y = M_1 \cup_{T^2} M_2$ where $M_1$ and $M_2$ are glued along their incompressible torus boundaries. Since $H_2(Y;\mathbb{Z}) \cong 0$, the Mayer–Vietoris sequence for this decomposition reads

$$0 \to \underbrace{H_1(T^2)}_{\cong \mathbb{Z}^2} \xrightarrow{i_*} H_1(M_1) \oplus H_1(M_2) \xrightarrow{j_*} \underbrace{H_1(Y)}_{\cong \mathbb{Z}/p^e\mathbb{Z}} \to 0.$$

Each map $H_1(T^2) \to H_1(M_i)$ induced by inclusion has rank 1, so $b_1(M_k) \geq 1$ for each $k$, but examining the sequence over $\mathbb{Q}$ shows that $b_1(M_1) + b_1(M_2) = 2$ and so in fact $b_1(M_k) = 1$. This means that we can write

$$H_1(M_k;\mathbb{Z}) \cong \mathbb{Z} \oplus T_k \qquad (k = 1, 2)$$

for some torsion groups $T_k$.

The nonzero elements of $T_1 \oplus T_2$ cannot lie in the image of the injective map $i_*$, since that image is free abelian, so by exactness we have an injection

$$j_*|_{T_1 \oplus T_2} : T_1 \oplus T_2 \hookrightarrow H_1(Y) \cong \mathbb{Z}/p^e\mathbb{Z}.$$

But $\mathbb{Z}/p^e\mathbb{Z}$ cannot be written as a direct sum of two nontrivial groups, so we conclude that one of the $T_k$ must be zero; without loss of generality we label $M_1$ and $M_2$ so that $T_1 = 0$. Then $T_2$ might be nonzero, but it does inject into $\mathbb{Z}/p^e\mathbb{Z}$, so we must have $T_2 \cong \mathbb{Z}/p^f\mathbb{Z}$ where $0 \leq f \leq e$. □

**Lemma 3.2.** *Suppose that $Y$ is $\mathrm{SL}(2,\mathbb{C})$-reducible and that $H_1(Y)$ is a finite cyclic group, but that $Y$ is not a lens space. Then $Y$ has an incompressible torus.*

*Proof.* Since $Y$ is $\mathrm{SL}(2,\mathbb{C})$-reducible it cannot be hyperbolic [CS83, Proposition 3.1.1]. An $\mathrm{SL}(2,\mathbb{C})$-reducible rational homology sphere is also $\mathrm{SU}(2)$-abelian, so if $Y$ is not a lens space then it can only be Seifert fibered with $H_1(Y)$ finite if its base orbifold is either $S^2(2,4,4)$ or $S^2(3,3,3)$ [SZ22a, Theorem 1.2]. We adapt the proof of [GSZ23, Lemma 8.2] to rule these out: if

$$Y \cong S^2((2,\beta_1),(4,\beta_2),(4,\beta_3))$$

then there is a surjection

$$H_1(Y) \cong \operatorname{coker} \begin{pmatrix} 2 & 0 & 0 & \beta_1 \\ 0 & 4 & 0 & \beta_2 \\ 0 & 0 & 4 & \beta_3 \\ 1 & 1 & 1 & 0 \end{pmatrix} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$$

given by sending the respective generators to $(1,2)$, $(1,1)$, $(0,1)$, and $(0,0)$, while if

$$Y = S^2((3,\beta_1),(3,\beta_2),(3,\beta_3))$$

then we have a surjection

$$H_1(Y) \cong \operatorname{coker} \begin{pmatrix} 3 & 0 & 0 & \beta_1 \\ 0 & 3 & 0 & \beta_2 \\ 0 & 0 & 3 & \beta_3 \\ 1 & 1 & 1 & 0 \end{pmatrix} \twoheadrightarrow \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$$

sending the generators to $(1,0)$, $(0,1)$, $(2,2)$, and $(0,0)$. In either case the first homology can't be cyclic, so this rules out all of the Seifert fibered possibilities, and now we conclude by the geometrization theorem that $Y$ must be toroidal. □

Supposing that $Y = M_1 \cup M_2$ as in Lemma 3.1 is SL$(2,\mathbb{C})$-reducible, we wish to arrange for the manifolds $M_1$ and $M_2$ to be as simple as possible. The following theorem of Rong will help us achieve this goal:

**Theorem 3.3** ([Ron92, Theorem 3.9])**.** *Suppose we have an infinite sequence of closed, oriented 3-manifolds and degree-1 maps between them, of the form*

$$Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} Y_3 \xrightarrow{f_3} \cdots .$$

*Then the map $f_i$ is a homotopy equivalence for all sufficiently large $i$.*

We will also use the following lemma.

**Lemma 3.4.** *Let $M_1$ and $M_2$ be compact 3-manifolds with incompressible torus boundary, and form*

$$Y = M_1 \cup_{T^2} M_2$$

*by gluing them according to some diffeomorphism $\partial M_1 \xrightarrow{\cong} \partial M_2$. If $Y$ is SL$(2,\mathbb{C})$-reducible, and if $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ where $n \geq 1$ does not have the form $n = 2e$ for some odd $e \geq 3$, then each of $M_1$, $M_2$, and $Y$ are irreducible.*

*Proof.* If $Y$ were reducible then it would be a connected sum, say $Y \cong Z \# Z'$, since it does not have the same homology as $S^1 \times S^2$. Both $Z$ and $Z'$ must be SL$(2,\mathbb{C})$-reducible if $Z\#Z'$ is, so neither one is a homology 3-sphere by Theorem 1.1. Now if neither $H_1(Z)$ nor $H_1(Z')$ were 2-torsion, then we could find an irreducible representation

$$\pi_1(Z\#Z') \cong \pi_1(Z) * \pi_1(Z') \twoheadrightarrow H_1(Z) * H_1(Z') \to \mathrm{SL}(2,\mathbb{C}),$$

exactly as in [GSZ23, Theorem 1.5]. Thus without loss of generality $H_1(Z)$ is 2-torsion, and

$$H_1(Y) \cong H_1(Z) \# H_1(Z') \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus r} \oplus H_1(Z')$$

for some $r \geq 1$. Since $H_1(Y)$ is cyclic we conclude that $r = 1$ and that $H_1(Z')$ is cyclic of odd order $e > 1$. But then $|H_1(Y)| = 2e$, and we have assumed that this is not the case, so $Y$ must be irreducible after all.

Now that we know $Y$ to be irreducible, we suppose that one of the $M_i$ is reducible, say $M_1$ without loss of generality. Then we can write $M_1 \cong W \# M_1'$, where $W \not\cong S^3$ is closed and $\partial M_1' \cong T^2$ is incompressible. In this case $Y \cong W \# (M_1' \cup_{T^2} M_2)$ is a nontrivial connected sum, because $M_1' \cup_{T^2} M_2$ is a closed toroidal manifold and hence not $S^3$. This contradicts the irreducibility of $Y$, so $M_1$ must have been irreducible after all. □

**Proposition 3.5.** *Suppose that there exists a toroidal,* $\mathrm{SL}(2,\mathbb{C})$*-reducible 3-manifold with* $H_1(Y;\mathbb{Z}) \cong \mathbb{Z}/p^e\mathbb{Z}$*, where* $p^e$ *is a prime power. Then there is such an irreducible 3-manifold* $Y$ *with the following additional properties: if we write*

$$Y = M_1 \cup_{T^2} M_2$$

*as in Lemma 3.1, and if* $\lambda_i$ *is the rational longitude of* $M_i$*, then*

(1) *both* $M_1$ *and* $M_2$ *are irreducible and boundary-incompressible,*
(2) $H_1(M_1) \cong \mathbb{Z}$*,*
(3) $M_2(\lambda_1)$ *is a lens space of order* $p^e$*,*
(4) *and either*
   (a) $\lambda_2$ *is nullhomologous,* $M_1(\lambda_2)$ *is a lens space of order* $p^e$*,* $H_1(M_2) \cong \mathbb{Z}$*, and* $\Delta(\lambda_1, \lambda_2) = p^e$*;*
   (b) *or* $\lambda_2$ *has order* $p^k$ *for some* $k$ *with* $1 \le k < e$*, and* $\Delta(\lambda_1, \lambda_2)$ *divides* $p^{e-k}$*.*

*Proof.* The irreducibility of $Y$ and of the $M_i$ is Lemma 3.4, so we will not discuss it further.

Let $Y_0$ be a toroidal, $\mathrm{SL}(2,\mathbb{C})$-reducible 3-manifold with $H_1(Y_0;\mathbb{Z}) \cong \mathbb{Z}/p^e\mathbb{Z}$. Supposing that the claimed $Y$ does not exist, we will inductively build an infinite sequence of 3-manifolds and degree-1 maps between them, of the form

$$Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} \dots,$$

such that the $f_i$ are not homotopy equivalences, and this will contradict Theorem 3.3.

Given $Y_i$, which is toroidal with $H_1(Y_i;\mathbb{Z}) \cong \mathbb{Z}/p^e\mathbb{Z}$, we use Lemma 3.1 to write

$$Y_i \cong M_1^i \cup_{T^2} M_2^i$$

with $H_1(M_1^i) \cong \mathbb{Z}$. Letting $\lambda_j^i$ be the rational longitude of $M_j^i$ for $j = 1, 2$, it follows that $\lambda_1^i$ has order 1, so there is a degree-1 map that pinches $M_1$ onto a solid torus while sending $\lambda_1^i$ to a longitude of that solid torus. (This is a classically known construction, but see [GSZ23, Proposition 4.2] for details.) This map extends to a degree-1 map

$$p_i : Y_i \to M_2^i(\lambda_1^i),$$

which is not a homotopy equivalence because the incompressibility of the separating torus $T^2$ guarantees that $\lambda_1^i$ is a nontrivial element of the kernel of $(p_1)_* : \pi_1(Y_i) \to \pi_1(Y_{i+1})$. Then $M_2^i(\lambda_1^i)$ satisfies $H_1(M_2^i(\lambda_1^i)) \cong H_1(Y_i) \cong \mathbb{Z}/p^e\mathbb{Z}$, since we have constructed it from $Y_i$ by replacing the homology solid torus $M_1^i$ with an actual solid torus in a way that preserves longitudes. If it is not a lens space then Lemma 3.2 says that it must be toroidal, so we let $Y_{i+1} = M_2^i(\lambda_1^i)$ and $f_i = p_i : Y_i \to Y_{i+1}$, and we continue to the next iteration.

If we have reached this point then $M_2^i(\lambda_1^i)$ must be a lens space of order $p^e$, and we now consider the rational longitude $\lambda_2^i$ of $M_2^i$.

Supposing for now that $\lambda_2^i$ has order 1, then Lemma 3.1 says that $H_1(M_2^i) \cong \mathbb{Z} \oplus \mathbb{Z}/p^f\mathbb{Z}$ for some $f \le e$, with the peripheral subgroup $H_1(\partial M_2^i)$ generating the $\mathbb{Z}$ summand since

$M_2^i$ is the complement of a nullhomologous knot. If we choose peripheral elements $\mu_1^i$ and $\mu_2^i$ dual to the longitudes $\lambda_1^i$ and $\lambda_2^i$, then the $\mu_j^i$ generate the $\mathbb{Z}$ summands of each $H_1(M_j^i)$. Dropping the "$i$" superscripts for now and writing the gluing map $\partial M_1^i \xrightarrow{\cong} \partial M_2^i$ in the form

$$\mu_1 \sim \mu_2^a \lambda_2^b$$
$$\lambda_1 \sim \mu_2^c \lambda_2^d$$

with $ad - bc = \pm 1$, we then compute that

$$\Delta(\lambda_1, \lambda_2) = \Delta(\mu_2^c \lambda_2^d, \lambda_2) = |c|,$$

that $\lambda_2^{\pm 1} \sim \mu_1^{-c} \lambda_1^a$, and that

$$\mathbb{Z}/p^e\mathbb{Z} \cong H_1(Y_i) \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\substack{(1,0)\sim(0,a) \\ (0,0)\sim(0,c)}} \oplus \mathbb{Z}/p^f\mathbb{Z} \cong \mathbb{Z}/c\mathbb{Z} \oplus \mathbb{Z}/p^f\mathbb{Z},$$

so $(\Delta(\lambda_1, \lambda_2), p^f) = (|c|, p^f)$ must be either $(1, p^e)$ or $(p^e, 1)$.

Now since $\lambda_2$ has order 1, there is again a degree-1 map pinching $M_2^i$ to a solid torus and sending $\lambda_2$ to a longitude of the solid torus; again this map

$$q_i : Y_i \to M_1^i(\lambda_2)$$

cannot be a homotopy equivalence. Then $M_1^i(\lambda_2)$ is $\mathrm{SL}(2, \mathbb{C})$-reducible, with homology

$$H_1(M_1^i(\lambda_2)) \cong \frac{H_1(M_1^i)}{\lambda_2} \cong \frac{H_1(M_1^i)}{\mu_1^{-c}\lambda_1^a} \cong \mathbb{Z}/|c|\mathbb{Z},$$

so we have two cases depending on the value of $|c| = \Delta(\lambda_1, \lambda_2)$:

(1) if $|c| = 1$ then $H_1(M_1^i(\lambda_2)) = 0$, so $M_1^i(\lambda_2)$ must be $S^3$ [Zen18];
(2) while if $|c| = p^e$ then $H_1(M_1^i(\lambda_2)) \cong \mathbb{Z}/p^e\mathbb{Z}$.

We use [GSZ23] to rule out the first case as follows: let $Z_1 = M_1^i(\lambda_2) \cong S^3$ and $Z_2 = M_2^i(\lambda_1)$, which is a lens space; the cores $K_1 \subset Z_1$ and $K_2 \subset Z_2$ of these fillings are nullhomologous, since their meridians $\mu_1 = \lambda_2$ and $\mu_2 = \lambda_1$ are at distance one from their longitudes $\lambda_1$ and $\lambda_2$, and their exteriors $M_1 \cong Z_1 \setminus N(K_1)$ and $M_2 \cong Z_2 \setminus N(K_2)$ are irreducible and boundary-incompressible. We form $Y_i$ by splicing the exteriors of $K_1$ and $K_2$, gluing the meridian of one to the longitude of the other and vice versa, and so [GSZ23, Proposition 9.8] gives us an irreducible representation $\rho : \pi_1(Y_i) \to \mathrm{SU}(2)$, contradicting the assumption that $Y_i$ was $\mathrm{SL}(2, \mathbb{C})$-reducible.

We must therefore be in case (2) above, meaning that $(|c|, p^f) = (p^e, 1)$ and that

$$H_1(M_2^i) \cong \mathbb{Z} \oplus \mathbb{Z}/p^f\mathbb{Z} \cong \mathbb{Z}.$$

If $M_1^i(\lambda_2)$ is a lens space of order $p^e$, then $Y_i$ is the desired $Y$ and we are done. Otherwise $\lambda_2$ has order 1 and $H_1(M_1^i(\lambda_2)) \cong \mathbb{Z}/p^e\mathbb{Z}$, but $M_1^i(\lambda_2)$ is not a lens space, and then it must be toroidal by Lemma 3.2. We let $Y_{i+1} = M_1^i(\lambda_2)$ and $f_i = q_i : Y_i \to Y_{i+1}$, and we continue to the next iteration.

The only case left to consider is where $M_2^i(\lambda_1^i)$ is a lens space of order $p^e$, but $\lambda_2^i$ is not nullhomologous in $M_2^i$; since it is a torsion element of $H_2(M_2^i) \cong \mathbb{Z} \oplus \mathbb{Z}/p^f\mathbb{Z}$ where $f \le e$, it must have order $p^k$, where $1 \le k \le f \le e$.

Identifying $H_2(M_2^i) \cong \mathbb{Z} \oplus \mathbb{Z}/p^f\mathbb{Z}$, we can choose a generator of the torsion summand so that the peripheral elements of $M_2^i$ have the form

$$\mu_2 = (a, b), \qquad \lambda_2 = (0, p^{f-k}).$$

Then as above we have a degree-1 map $Y_i \to M_2^i(\lambda_1)$, which preserves $H_1$ because it replaces the homology solid torus $M_1^i$ with an actual solid torus in a longitude-preserving way. If $\lambda_1 \sim \mu_2^c \lambda_2^d$ for some integers $c$ and $d$, then

$$\mathbb{Z}/p^e\mathbb{Z} \cong H_1(M_2^i(\lambda_1)) \cong \frac{\mathbb{Z} \oplus \mathbb{Z}/p^f\mathbb{Z}}{c(a,b) + d(0, p^{f-k}) = 0} \cong \operatorname{coker} \begin{pmatrix} ca & cb + dp^{f-k} \\ 0 & p^f \end{pmatrix},$$

so by comparing orders we have

$$|ac| p^f = p^e,$$

or equivalently $|ac| = p^{e-f}$. In particular, if $k = e$ then these are each equal to $f$, which is sandwiched between $k$ and $e$, so we have $|ac| = 1$ and thus

$$\mu_2 = (\pm 1, b), \qquad \lambda_2 = (0, 1)$$

as elements of $\mathbb{Z} \oplus \mathbb{Z}/p^f\mathbb{Z}$. But then these elements span a 2-dimensional subspace of $H_1(M_2^i; \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, contradicting the "half lives half dies" theorem which says that the inclusion-induced map

$$H_1(\partial M_2^i; \mathbb{F}) \to H_1(M_2^i; \mathbb{F})$$

has rank 1 over any field $\mathbb{F}$. We must therefore have a strict inequality $k < e$, and moreover

$$\Delta(\lambda_1, \lambda_2) = \Delta(\mu_2^c \lambda_2^d, \lambda_2) = |c|$$

divides $p^{e-f}$ and hence $p^{e-k}$ as claimed. □

## 4. Toroidal manifolds with essential rational longitudes

Having proved Proposition 3.5 for manifolds with homology $\mathbb{Z}/p^e\mathbb{Z}$ in general, we now specialize to $p^e = 4$ in order to rule out the last case of that proposition. We accomplish this by simplifying our toroidal 3-manifold via [GSZ23, Proposition 1.9], asserting that a compact 3-manifold with torus boundary and a rational longitude of order 2 admits a degree-1 map onto the twisted $I$-bundle over the Klein bottle. We thus begin by describing the SU(2) representations of the latter manifold in the following lemma, which is a corrected version of the second half of [GSZ23, Proposition 4.4].

**Lemma 4.1.** *Let $N$ be the twisted $I$-bundle over the Klein bottle, with peripheral Seifert fiber $\sigma$ and rational longitude $\lambda$. Then every representation $\rho : \pi_1(N) \to \mathrm{SU}(2)$ is conjugate to one satisfying*

$$(\rho(\sigma), \rho(\lambda)) = (e^{i\alpha}, \pm 1) \quad \text{or} \quad (\rho(\sigma), \rho(\lambda)) = (-1, e^{i\beta})$$

*for arbitrary $\alpha, \beta \in \mathbb{R}/2\pi\mathbb{Z}$, and all such pairs are realized by some $\rho$. The image of $\rho$ is non-abelian if and only if $\rho(\lambda) \neq \pm 1$.*

*Proof.* Following [GSZ23, Proposition 4.4], we write

$$\pi_1(N) \cong \langle a, b \mid aba^{-1} = b^{-1} \rangle,$$

with $\sigma = a^2$ and $\lambda = b$. Up to conjugacy any $\rho : \pi_1(N) \to \mathrm{SU}(2)$ satisfies $\rho(\sigma) = e^{i\alpha}$ and $\rho(\lambda) = e^{i\beta}$ for some $\alpha, \beta \in \mathbb{R}/2\pi\mathbb{Z}$, so we will assume throughout the proof that all representations $\rho$ have this form.

We first suppose that $\rho : \pi_1(N) \to \mathrm{SU}(2)$ has abelian image, and we claim that it must then satisfy $\rho(\lambda) = \pm 1$. Indeed, if $\rho(\lambda) = e^{i\beta}$ were different from $\pm 1$, then $\rho(a)$ would have to lie in the unique $U(1)$ subgroup through $\rho(\lambda)$, so in fact we could write $\rho(a) = e^{it}$ for some $t$. But then the relation $\rho(aba^{-1}) = \rho(b^{-1})$ would become $e^{i\beta} = (e^{i\beta})^{-1}$, so $\rho(\lambda) = e^{i\beta}$ must have been $\pm 1$ after all, a contradiction. At the same time, given any $\alpha$ we can define a representation $\rho : \pi_1(Y) \to \mathrm{SU}(2)$ with abelian image by

$$\rho(a) = e^{i\alpha/2}, \qquad \rho(b) = \pm 1,$$

and then we have $\rho(\sigma) = \rho(a^2) = e^{i\alpha}$ and $\rho(\lambda) = \rho(b) = \pm 1$, so all pairs $(\rho(\sigma), \rho(\lambda)) = (e^{i\alpha}, \pm 1)$ are realized by abelian representations.

Now suppose that $\rho : \pi_1(N) \to \mathrm{SU}(2)$ has non-abelian image. Since the element $\sigma = a^2$ is central in $\pi_1(N)$, its image $\rho(\sigma)$ must commute with the entire image of $\rho$, so then $\rho(\sigma) = \pm 1$. If we had $\rho(\sigma) = +1$ then its square root $\rho(a)$ would have to be $\pm 1$, but this is not possible if $\rho$ has non-abelian image, so in fact $\rho(\sigma) = -1$. Up to conjugacy this means that $\rho(a) = j$, and then the relation

$$j\rho(b)j^{-1} = \rho(a)\rho(b)\rho(a)^{-1} = \rho(b)^{-1}$$

implies that the $j$-component of $\rho(b)$ is zero, so we can conjugate again by something of the form $\cos(\theta) + \sin(\theta)j$ to maintain the relation $\rho(a) = j$ while recovering our preferred form $\rho(b) = e^{i\beta}$. Any such choice of $\beta$ gives rise to a representation $\rho$ with $\rho(\sigma) = -1$ and $\rho(\lambda) = \rho(b) = e^{i\beta}$, and it has non-abelian image precisely when $j$ does not commute with $e^{i\beta}$, i.e., when $e^{i\beta} \neq \pm 1$. □

**Remark 4.2.** The abelian representations with $\rho(\sigma) \neq 1$ in Lemma 4.1 were mistakenly omitted from [GSZ23, Proposition 4.4], though this omission does not affect its application in [GSZ23]. In the proof of that proposition we had claimed that if $\rho(a) = e^{js}$ for some $s \notin \pi\mathbb{Z}$, then "the relation $\rho(aba^{-1}) = \rho(b^{-1})$ implies that $\rho(a) = \pm j$ and that $\rho(b)$ has zero $j$-component," when in fact another possibility is that $s$ is arbitrary and $\rho(b) = \pm 1$.

We now begin to address case (4b) of Proposition 3.5 for manifolds $Y$ with $H_1(Y) \cong \mathbb{Z}/4\mathbb{Z}$. The following is a special case, in which we take the simplest possible choice of $M_2$ whose rational longitude has order 2 and consider a very specific gluing map $\partial M_1 \xrightarrow{\cong} \partial M_2$.

**Proposition 4.3.** *Let $N$ denote the twisted $I$-bundle over the Klein bottle, with peripheral Seifert fiber $\mu_2$ and rational longitude $\lambda_2$. Form a closed 3-manifold*

$$Y = M_1 \cup_{T^2} N$$

*where*

1. *$H_1(Y; \mathbb{Z})$ is finite cyclic and $H_1(M_1; \mathbb{Z}) \cong \mathbb{Z}$;*
2. *$M_1$ is irreducible and has incompressible torus boundary, with longitude $\lambda_1$;*
3. *and the gluing map identifies $\lambda_1 \sim \mu_2^{\pm 1}\lambda_2$.*

*Then there is a representation*

$$\rho : \pi_1(Y) \to \mathrm{SU}(2)$$

*with non-abelian image.*

*Proof.* We write $\lambda_1 \sim \mu_2^\epsilon \lambda_2$ for some $\epsilon \in \{\pm 1\}$ and choose $\mu_1 \subset \partial M_1$ to be the peripheral curve satisfying $\mu_1 \sim \lambda_2^\epsilon$, so that

$$\begin{cases} \mu_1 \sim \quad \lambda_2^\epsilon \\ \lambda_1 \sim \mu_2^\epsilon \lambda_2 \end{cases} \iff \begin{cases} \mu_2 \sim \mu_1^{-1} \lambda_1^\epsilon \\ \lambda_2 \sim \mu_1^\epsilon. \end{cases}$$

Then we have

$$\Delta(\mu_1, \lambda_1) = \Delta(\lambda_2^\epsilon, \mu_2^\epsilon \lambda_2) = 1,$$

so $\mu_1$ is dual to $\lambda_1$ as curves in $\partial M_1$, and in particular $\mu_1$ must generate $H_1(M_1) \cong \mathbb{Z}$. This means that

$$Y_1 = M_1(\mu_1)$$

is an integer homology sphere. We let $K_1 \subset Y_1$ be the core of this filling, with exterior $M_1 \cong Y_1 \setminus N(K_1)$.

We first suppose that $Y_1$ is not SU(2)-abelian. Then there is a representation

$$\rho_{Y_1} : \pi_1(Y_1) \to \mathrm{SU}(2)$$

with non-abelian image, and this gives rise to a non-abelian

$$\rho_{M_1} : \pi_1(M_1) \twoheadrightarrow \frac{\pi_1(M_1)}{\langle\!\langle \mu_1 \rangle\!\rangle} \cong \pi_1(Y_1) \xrightarrow{\rho_{Y_1}} \mathrm{SU}(2)$$

satisfying $\rho_{M_1}(\mu_1) = 1$. Now by Lemma 4.1 we can take an abelian representation

$$\rho_N : \pi_1(N) \to \mathrm{SU}(2)$$

satisfying $\rho_N(\lambda_2) = 1$ and $\rho_N(\mu_2) = \rho_{M_1}(\lambda_1^\epsilon)$, and since $\rho_{M_1}$ and $\rho_N$ satisfy

$$\rho_N(\mu_2) = \rho_{M_1}(\mu_1^{-1}\lambda_1^\epsilon)$$
$$\rho_N(\lambda_2) = \rho_{M_1}(\mu_1^\epsilon),$$

they glue together to give a representation $\rho : \pi_1(Y) \to \mathrm{SU}(2)$. The image of $\rho$ contains that of $\rho_{M_1}$, so it is non-abelian and in this case we are done.

For the remainder of the proof we can suppose that $Y_1$ is SU(2)-abelian. Since the exterior $M_1$ of $K_1 \subset Y_1$ is irreducible and boundary-incompressible, Theorem 2.3 provides a topologically embedded curve $\gamma$ in the cut-open pillowcase image

$$i^* X(Y_1, K_1) \subset \mathcal{C} = [0,\pi]_\alpha \times (\mathbb{R}/2\pi\mathbb{Z})_\beta$$

that is homologically nontrivial in $H_1(\mathcal{C};\mathbb{Z})$, and that satisfies

$$\gamma \subset (0,\pi) \times (\mathbb{R}/2\pi\mathbb{Z}) = \mathrm{int}(\mathcal{C}).$$

Now for each possible $\epsilon \in \{\pm 1\}$ and each $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we define a line segment $L_\theta^\epsilon$ from the $\alpha = 0$ component of $\partial\mathcal{C}$ to the $\alpha = \pi$ component of $\partial\mathcal{C}$ by setting

$$L_\theta^\epsilon = \{(\alpha, \epsilon(\alpha+\theta)) \mid 0 \le \alpha \le \pi\}.$$

See Figure 1. Each of the paths $L_\theta^\epsilon$ generates the relative homology $H_1(\mathcal{C}, \partial\mathcal{C};\mathbb{Z}) \cong \mathbb{Z}$, so the intersection pairing

$$H_1(\mathcal{C}) \times H_1(\mathcal{C}, \partial\mathcal{C}) \to \mathbb{Z}$$

sends $([\gamma], [L_\theta^\epsilon])$ to $\pm 1$, and as such the intersection

$$\gamma \cap L_\theta^\epsilon$$

must be nonempty.

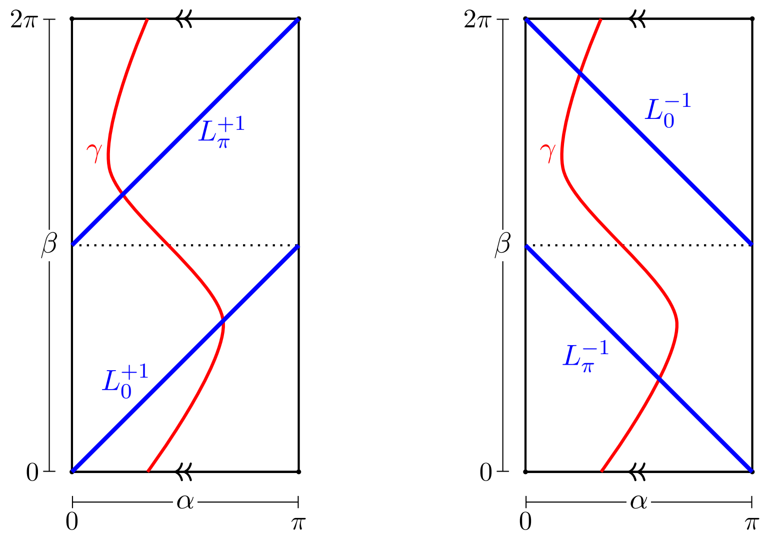


FIGURE 1. Left: the curve $\gamma$ meeting each of the paths $L_0^{+1}$ and $L_\pi^{+1}$ in the cut-open pillowcase. Right: same, but with $L_0^{-1}$ and $L_\pi^{-1}$.

Any point $(\alpha, \beta)$ of $\gamma \cap L_\theta^\epsilon$ corresponds to a representation

$$\begin{aligned} \rho_{M_1} : \pi_1(M_1) &\to \mathrm{SU}(2) \\ \mu_1 &\mapsto e^{i\alpha} \\ \lambda_1 &\mapsto e^{i\beta}, \end{aligned}$$

satisfying $\rho_{M_1}(\mu_1^\epsilon) = e^{i\cdot\epsilon\alpha}$ and

$$\rho_{M_1}(\mu_1^{-1}\lambda_1^\epsilon) = e^{i(-\alpha+\epsilon\beta)} = e^{i\theta}.$$

We cannot have $\alpha = 0$ or $\alpha = \pi$, because $\gamma$ does not contain any such points, and so $\rho_{M_1}(\mu_1^\epsilon) = e^{i\cdot\epsilon\alpha}$ is not equal to $\pm 1$. In other words, for each $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we have found a representation $\rho_{M_1} : \pi_1(M_1) \to \mathrm{SU}(2)$ satisfying

$$\rho_{M_1}(\mu_1^{-1}\lambda_1^\epsilon) = e^{i\theta}, \qquad\qquad \rho_{M_1}(\mu_1^\epsilon) = e^{i\cdot\epsilon\alpha} \neq \pm 1.$$

At this point, we specialize to $\theta = \pi$ and use Lemma 4.1 to identify a representation $\rho_N : \pi_1(N) \to \mathrm{SU}(2)$ satisfying

$$\begin{aligned} \rho_N(\mu_2) &= e^{i\theta} = -1 \\ \rho_N(\lambda_2) &= \rho_{M_1}(\mu_1^\epsilon) \neq \pm 1, \end{aligned}$$

which must necessarily be non-abelian. Since $\mu_2 \sim \mu_1^{-1}\lambda_1^\epsilon$ and $\lambda_2 \sim \mu_1^\epsilon$, and we have arranged that $\rho_N(\mu_2) = \rho_{M_1}(\mu_1^{-1}\lambda_1^\epsilon)$ and $\rho_N(\lambda_2) = \rho_{M_1}(\mu_1^\epsilon)$, the representations $\rho_{M_1}$ and $\rho_N$ glue together to give us a representation

$$\rho : \pi_1(Y) \to \mathrm{SU}(2),$$

whose image is non-abelian because it contains the non-abelian image of $\rho_N$. □

**Proposition 4.4.** *Assume the hypotheses of Proposition 3.5. If $p^e = 4$, then case* (4b) *of that proposition does not occur. In other words, case* (4a) *applies instead: the rational* *longitude* $\lambda_2 \subset \partial M_2$ *is nullhomologous,* $M_1(\lambda_2)$ *is a lens space of order* $p^e = 4$, $H_1(M_2) \cong \mathbb{Z}$, *and* $\Delta(\lambda_1, \lambda_2) = p^e = 4$.

*Proof.* Supposing that we find ourselves in case (4b), the rational longitude $\lambda_2$ must have order exactly 2, and the distance

$$\Delta(\lambda_1, \lambda_2)$$

divides 2. Since $\lambda_2$ has order 2, we apply [GSZ23, Proposition 1.9] to produce a degree-1 map

$$M_2 \to N$$

that preserves rational longitudes, where $N$ is the twisted $I$-bundle over the Klein bottle. This gives rise to a sequence of rational-longitude-preserving, degree-1 maps

$$Y = M_1 \cup_{T^2} M_2 \to M_1 \cup_{T^2} N \to N(\lambda_1), \tag{4.1}$$

where we first pinch $M_2$ to $N$ and then $M_1$ to a solid torus.

It follows from (4.1) that $M_1 \cup_{T^2} N$ and $N(\lambda_1)$ are both $\mathrm{SL}(2,\mathbb{C})$-reducible, with first homology a quotient of $\mathbb{Z}/4\mathbb{Z}$. In fact, every Dehn filling of $N$ has homology of order at least 4 and so we must have

$$H_1(N(\lambda_1); \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z},$$

which implies in turn that

$$H_1(M_1 \cup_{T^2} N) \cong \mathbb{Z}/4\mathbb{Z}.$$

We can also read from the proof of [GSZ23, Proposition 4.4] that if $\mu_2$ is the Seifert fiber slope on $\partial N$, then

$$\lambda_1 \sim \mu_2^{\pm 1} \lambda_2$$

for some choice of sign, since otherwise $N(\lambda_1)$ cannot be $\mathrm{SL}(2,\mathbb{C})$-reducible with homology $\mathbb{Z}/4\mathbb{Z}$. But now Proposition 4.3 provides us with a non-abelian representation

$$\pi_1(M_1 \cup_{T^2} N) \to \mathrm{SU}(2),$$

so $M_1 \cup_{T^2} N$ cannot be $\mathrm{SL}(2,\mathbb{C})$-reducible and we have a contradiction. ☐

## 5. Gluing 3-manifolds with nullhomologous rational longitudes

With Propositions 3.5 and 4.4 in mind, our main goal in this lengthy section is to prove the following theorem.

**Theorem 5.1.** *Let* $Y = M_1 \cup_{T^2} M_2$ *be an irreducible 3-manifold with* $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$, *where*

(1) $M_1$ *and* $M_2$ *are irreducible, with incompressible torus boundaries;*
(2) $H_1(M_1; \mathbb{Z}) \cong H_1(M_2; \mathbb{Z}) \cong \mathbb{Z}$;
(3) *and if* $\lambda_j \subset \partial M_j$ *denotes the longitude of* $M_j$, *then both* $M_1(\lambda_2)$ *and* $M_2(\lambda_1)$ *are lens spaces of order* 4.

*Then there is a representation* $\rho : \pi_1(Y) \to \mathrm{SU}(2)$ *with non-abelian image.*

We will first deduce Theorem 1.4 from Theorem 5.1 before going on to prove the latter.

*Proof of Theorem 1.4.* Let $Y$ be closed and orientable, with $H_1(Y) \cong \mathbb{Z}/4\mathbb{Z}$, and suppose that $Y$ is not homeomorphic to a lens space. Then Lemma 3.2 says that $Y$ must have an incompressible torus. Proposition 3.5 subsequently produces another such $Y'$ that is both toroidal and $\mathrm{SL}(2,\mathbb{C})$-reducible, and by Proposition 4.4 it satisfies the hypotheses of Theorem 5.1. But then Theorem 5.1 gives us a representation

$$\rho : \pi_1(Y') \to \mathrm{SU}(2) \hookrightarrow \mathrm{SL}(2,\mathbb{C})$$

with irreducible image, and we have a contradiction. □

We begin in §5.1 by gathering some facts about the pillowcase images of knots with lens space surgeries. In §5.2 we specialize to the case where the lens space surgery has slope 4, showing that we can form $Y$ up to an overall orientation reversal by a very specific gluing of knot complements and describing essential curves in the pillowcase images of these knots. Then in §5.3 we begin to show that such $Y$ admit non-abelian representations $\rho : \pi_1(Y) \to \mathrm{SU}(2)$, using these essential curves to find the desired $\rho$ when neither curve passes through the point $(\frac{\pi}{2}, 0)$ in the pillowcase. In §5.4 we apply instanton knot homology to settle the remaining case, where at least one curve contains $(\frac{\pi}{2}, 0)$, and this completes the proof.

5.1. **Pillowcase images of knots with lens space surgeries.** Motivated by Proposition 3.5, we prove some general facts about the pillowcase images of knots in homology spheres that admit lens space surgeries. We begin with the observation that a cyclic surgery forces the pillowcase image to avoid lines of the corresponding slope in the pillowcase.

**Lemma 5.2.** *Let $K$ be a knot in an integer homology sphere $Y$, with complement $E_K$, and with meridian and longitude $\mu, \lambda \subset \partial E_K$. Suppose that $Y_{r/s}(K)$ is a lens space for some relatively prime $r, s$ with $r \neq 0$ and $s \geq 1$. Then any representation*

$$\rho : \pi_1(E_K) \to \mathrm{SU}(2)$$

*such that $\rho(\mu^r\lambda^s) = \pm 1$ must have finite cyclic image and satisfy $\rho(\lambda) = 1$. In other words, the intersection*

$$i^*X^{irr}(Y,K) \cap \{(\alpha,\beta) \mid r\alpha + s\beta \equiv 0 \pmod{\pi}\} \subset X(T^2)$$

*is empty.*

*Proof.* Let $\rho$ be such a representation. Then $\mathrm{ad}\,\rho : \pi_1(E_K) \to \mathrm{SO}(3)$ sends $\mu^r\lambda^s$ to 1, so it descends to a representation

$$\pi_1(Y_{r/s}(K)) \cong \frac{\pi_1(E_K)}{\langle\langle \mu^r\lambda^s \rangle\rangle} \to \mathrm{SO}(3)$$

with the same image, and then $\mathrm{ad}\,\rho$ must have finite cyclic image because $\pi_1(Y_{r/s}(K))$ is finite cyclic. The image of $\rho$ is therefore a finite subgroup $G \subset \mathrm{SU}(2)$ whose image under $\mathrm{ad} : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is cyclic, and it follows that $G$ is finite cyclic as well. Now since $\rho$ has abelian image it factors through $H_1(E_K)$, in which $[\lambda] = 0$, and so $\rho(\lambda) = 1$ as claimed. □

See Figure 2 for an illustration in the case where $\frac{r}{s} = 4$.

We can also show that the image $i^*X^{\mathrm{irr}}(Y,K)$ avoids an open neighborhood of each point $(\frac{k\pi}{|r|}, 0)$. The following generalizes [GSZ23, Lemma 10.4], which addressed the case where the slope $\frac{r}{s}$ has numerator an odd prime.

**Proposition 5.3.** *Let $K$ be a knot in a homology sphere $Y$, with irreducible and boundary-incompressible complement. Suppose that $Y_{r/s}(K)$ is a lens space, where $r \neq 0$ and $s \geq 1$ are relatively prime. Then there is an open neighborhood of the set*

$$\{r\alpha + s\beta \equiv 0 \pmod{\pi}\} \subset X(T^2),$$

*including each point $(\frac{k\pi}{|r|}, 0)$ with $0 \leq k \leq |r|$, that is disjoint from the image $i^* X^{irr}(Y, K)$ of the* irreducible *representations of $\pi_1(E_K)$. In particular, this neighborhood intersects the pillowcase image $i^* X(Y, K)$ only along the line segment $\beta \equiv 0 \pmod{2\pi}$.*

*Proof.* If we cannot find such a neighborhood, then some $(\alpha_0, \beta_0)$ with $r\alpha_0 + s\beta_0 \in \pi\mathbb{Z}$ is a limit point of $i^* X^{\mathrm{irr}}(Y, K)$. There is then a sequence of irreducible representations $\pi_1(E_K) \to \mathrm{SU}(2)$ whose images in the pillowcase converge to $(\alpha_0, \beta_0)$, and since $R(Y, K)$ is compact, some subsequence converges in $R(Y, K)$ to a representation $\rho : \pi_1(E_K) \to \mathrm{SU}(2)$ with pillowcase coordinates $i^*[\rho] = (\alpha_0, \beta_0)$. By Lemma 5.2 it follows that $\rho$ has abelian image and that $\beta_0 = 0$, and then $(\alpha_0, \beta_0) = (\frac{k\pi}{|r|}, 0)$ for some integer $k$ with $0 \leq k \leq |r|$. In other words, up to conjugacy $\rho$ satisfies

$$\rho(\mu) = e^{i \cdot k\pi/|r|}, \qquad \rho(\lambda) = 1.$$

Given that $\rho$ is a reducible limit of irreducible representations, with $\rho(\mu) = e^{i \cdot k\pi/|r|}$, we know once again from [GSZ23, Lemma 3.2] that $k$ is neither 0 nor $|r|$. Heusener, Porti, and Suárez Peiró [HPSP01, Theorem 2.7] (cf. Klassen [Kla91]) moreover proved that the Alexander polynomial of $K \subset Y$ must satisfy

$$\Delta_K\left(e^{i \cdot 2k\pi/|r|}\right) = 0.$$

Thus by [BZ03, Theorem 8.21] the branched $|r|$-fold cyclic cover $\tilde{Y} = \Sigma_{|r|}(K)$ of $K \subset Y$ has $b_1(\tilde{Y}) > 0$. If $\tilde{K} \subset \tilde{Y}$ is the lift of the branch locus $K$, then its meridian $\mu_{\tilde{K}}$ is a lift of $\mu_K^{|r|}$ while the longitude $\lambda_{\tilde{K}}$ lifts $\lambda_K$, and so the $|r|$-fold covering $\tilde{Y} \setminus N(\tilde{K}) \to Y \setminus N(K)$ extends to an $|r|$-fold covering

$$\tilde{Y}_{\mathrm{sign}(r)/s}(\tilde{K}) \to Y_{r/s}(K).$$

But $Y_{r/s}(K)$ is a lens space of order $|r|$, so we must have $\tilde{Y}_{\pm 1/s}(\tilde{K}) \cong S^3$. In particular $\tilde{Y}$ is actually a homology sphere and we have a contradiction. □

Given a knot $K \subset Y$ with a lens space surgery, we will now show, following [Zen18], that under mild hypotheses the pillowcase image of $K$ must contain an essential simple closed curve in the twice-punctured pillowcase. This relies on a nonvanishing result for the instanton homology of the zero-surgery $Y_0(K)$, for which by [KM10] it will suffice to know that $Y_0(K)$ is irreducible. We thus prove the following generalization of part of [GSZ23, Proposition 6.2], referring the reader to [Sco83] for basic facts about Seifert fibered spaces.

**Proposition 5.4.** *Let $Y$ be a homology 3-sphere, and let $K \subset Y$ be a knot with irreducible, boundary-incompressible exterior. Suppose that $Y_{r/s}(K)$ is a lens space for some relatively prime integers $r$ and $s$, with $|r| \geq 2$ and $s \geq 1$. Then $Y_0(K)$ is irreducible.*

*Proof.* Suppose instead that $Y_0(K)$ is reducible. Then $K$ has a cyclic surgery of slope $\frac{r}{s}$ and a reducible surgery of slope 0, and the distance between these slopes is $\Delta(\frac{r}{s}, \frac{0}{1}) = |r| \geq 2$, so Boyer and Zhang [BZ98, Theorem 1.2(1)] proved that $E_K = Y \setminus N(K)$ is either a simple (i.e.,

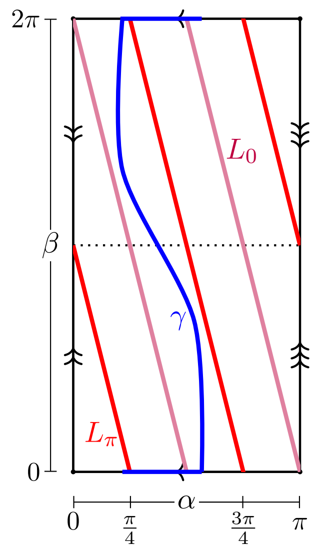


FIGURE 2. If $Y_4(K)$ is a lens space, then the pillowcase image $i^*X(Y,K)$ must avoid the lines $L_0 = \{4\alpha + \beta \equiv 0 \pmod{2\pi}\}$ and $L_\pi = \{4\alpha + \beta \equiv \pi \pmod{2\pi}\}$, except at the points where $\beta \equiv 0 \pmod{2\pi}$. The curve $\gamma \subset i^*X(Y,K)$ shown here, as provided by Proposition 5.5, only meets these lines at $(\frac{\pi}{4}, 0)$ and $(\frac{\pi}{2}, 0)$.

irreducible and atoroidal) Seifert fibered manifold or a cable on the twisted I-bundle over the Klein bottle. It cannot be the latter, because the Klein bottle would then represent a nonzero class in $H_2(Y; \mathbb{Z}/2\mathbb{Z}) = 0$ – it does not even separate its own tubular neighborhood, so there is a closed curve dual to it – so $E_K$ must be simple Seifert fibered.

Next, we claim that $E_K$ has orientable base orbifold. Indeed, the Seifert fibration of $E_K$ extends over any Dehn filling $E_K(\gamma)$ as long as $\gamma$ is not the Seifert fiber slope, so it extends over all but at most one of the surgeries $Y_{1/n}(K)$, where $n \in \mathbb{Z}$. If the base orbifold $\Sigma$ of some $Y_{1/n}(K)$ were non-orientable, then we could pull the Seifert fibration back along the orientation double cover $\tilde{\Sigma} \to \Sigma$ to get a double cover of $Y_{1/n}(K)$, which is impossible because $Y_{1/n}(K)$ is a homology sphere. Thus $\Sigma$ must be orientable, and the base orbifold of $E_K$ is orientable as well because it is $\Sigma$ minus an open disk.

We now show that $Y_0(K) \cong S^1 \times S^2$. If the longitude $\lambda$ of $K$ is not the Seifert fiber slope on $\partial E_K$, then the Seifert fibration on $E_K$ extends to $E_K(\lambda) \cong Y_0(K)$, and then $Y_0(K)$ is reducible (by assumption) and Seifert fibered but does not have the homology of $\mathbb{RP}^3 \# \mathbb{RP}^3$, so it must be $S^1 \times S^2$. Otherwise, if $\lambda$ is the Seifert fiber slope, then since the base orbifold of $E_K$ is orientable we know that $Y_0(K)$ will be a connected sum of several copies of $S^1 \times S^2$ and lens spaces [Hei74, Proposition 2], and then the only way that we can have $H_1(Y_0(K)) \cong \mathbb{Z}$ is if $Y_0(K) \cong S^1 \times S^2$.

To summarize, we now know that $E_K = Y \setminus N(K)$ is Seifert fibered, with orientable base orbifold; that $Y_{r/s}(K)$ is a lens space; and that $Y_0(K) \cong S^1 \times S^2$, so $E_K$ is also the exterior of a knot $K' \subset S^1 \times S^2$. From these facts, Baker, Buck, and Lecuona [BBL16, Theorem 1.18] proved that $K'$ must be either an $(a,b)$-torus knot or a $(2, \pm 1)$-cable of a torus knot, and $K'$ cannot be cabled because we know from the above application of [BZ98] that $E_K \cong E_{K'}$ is atoroidal. As $K'$ is a $(a,b)$-torus knot in $S^1 \times S^2$, which in [BBL16] means a cable of $S^1 \times \{\text{pt}\}$ with winding number $a$, its complement $E_{K'}$ has first homology $H_1(E_{K'}) \cong \mathbb{Z} \oplus \mathbb{Z}/a\mathbb{Z}$. But we also know that $H_1(E_K) \cong \mathbb{Z}$, so $|a| = 1$ and therefore $K'$ is isotopic to $S^1 \times \{\text{pt}\}$. Then $E_K \cong E_{K'}$ must be a solid torus, and this contradicts the incompressibility of $\partial E_K$, so $Y_0(K)$ must have been irreducible after all. □

**Proposition 5.5.** *Let $K$ be a knot in a homology sphere $Y$, with irreducible, boundary-incompressible exterior. Suppose that $Y_{r/s}(K)$ is a lens space for some relatively prime integers $r, s$ with $|r| \geq 2$ and $s \geq 1$. Then the pillowcase image $i^*X(Y,K) \subset X(T^2)$ does not contain either of the points $(0,\pi)$ and $(\pi,\pi)$, and moreover there is a topologically embedded, closed curve*

$$\gamma \subset i^*X(Y,K)$$

*that is homologically essential in the twice-punctured pillowcase $X(T^2)\setminus\{(0,\pi),(\pi,\pi)\}$, and that does not contain either $(0,0)$ or $(\pi,0)$.*

*Proof.* Let $E_K$ be the exterior of $K$. If there were a representation $\rho : \pi_1(E_K) \to \mathrm{SU}(2)$ corresponding to either $(\alpha,\beta) = (0,\pi)$ or $(\alpha,\beta) = (\pi,\pi)$ in the pillowcase, then it would satisfy $\rho(\mu) = \pm 1$ and $\rho(\lambda) = -1$, so that $\rho(\mu^r\lambda^s) = \pm 1$. Lemma 5.2 says that any such representation necessarily has $\rho(\lambda) = 1$, so the claimed $\rho$ cannot exist.

Now we observe that $Y_0(K)$ is irreducible by Proposition 5.4, and so [KM10, Theorem 7.21] says that $I^w_*(Y_0(K)) \neq 0$, where $w$ is the nonzero element of $H^2(Y_0(K);\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Since we have also shown that $i^*X(Y,K)$ avoids $(0,\pi)$ and $(\pi,\pi)$, we can apply [GSZ23, Proposition 3.1], which is really a slight generalization of results of [Zen18, §7], to find a homologically essential curve $\gamma \subset i^*X(Y,K)$.

We must now show that $\gamma$ can be arranged to avoid $(0,0)$ and $(\pi,0)$. To do so, we observe that since $Y_{r/s}(K)$ is a lens space and therefore SU(2)-abelian, we know by [GSZ23, Lemma 3.2] that neither $(0,0)$ nor $(\pi,0)$ is a limit point of the image $i^*X^{\mathrm{irr}}(Y,K)$ of the irreducible character variety of $K$. This means that each of these points has an open neighborhood in the pillowcase whose intersection with $i^*X(Y,K)$ consists only of the line $\beta \equiv 0 \pmod{2\pi}$, realized by reducible characters. There is therefore a deformation retraction of $i^*X(Y,K)$ taking it into the complement of these neighborhoods, just as in the proof of Theorem 2.3, and the image of $\gamma$ under this deformation retraction will be the desired curve, since it is still contained in $i^*X(Y,K)$ but does not pass through either $(0,0)$ or $(\pi,0)$. □

5.2. **Knots with cyclic surgeries of order 4.** We now apply the results of the preceding subsection to the case where the prime power in question is 4. We must first see how to decompose a toroidal manifold with homology $\mathbb{Z}/4\mathbb{Z}$ into a pair of knot exteriors glued along their boundaries.

**Proposition 5.6.** *Suppose that $Y = M_1 \cup_{T^2} M_2$, where $M_1$ and $M_2$ are compact oriented 3-manifolds with torus boundary satisfying*

$$H_1(Y) \cong \mathbb{Z}/4\mathbb{Z}, \qquad H_1(M_1) \cong H_1(M_2) \cong \mathbb{Z}.$$

*Then up to possibly reversing the orientation of $Y$, we can write $M_i = Y_i \setminus N(K_i)$, where each $Y_i$ is a homology sphere, the knot $K_i \subset Y_i$ has meridian $\mu_i$ and longitude $\lambda_i$, and the gluing map identifies*

$$\begin{aligned} \mu_1 &\sim \mu_2, \\ \lambda_1^{-1} &\sim \mu_2^4\lambda_2. \end{aligned} \tag{5.1}$$

*In particular, we have* $(Y_1)_4(K_1) \cong M_1(\lambda_2)$ *and* $(Y_2)_4(K_2) \cong M_2(\lambda_1)$.

*Proof.* We choose peripheral curves $\mu_i \subset \partial M_i$ that are dual to the longitudes $\lambda_i$, where if $\lambda_1$ is dual to $\lambda_2$ then we will take $\mu_1 = \lambda_2$ and $\mu_2 = \lambda_1$. Then we let

$$Y_i = M_i(\mu_i) \qquad (i = 1, 2),$$

so that each $Y_i$ is a homology sphere, and the core of each Dehn filling is a nullhomologous knot $K_i \subset Y_i$ with complement $M_i$. If we write the gluing map $\partial M_1 \xrightarrow{\cong} \partial M_2$ as

$$\begin{aligned} \mu_1 &\sim \mu_2^a \lambda_2^b \\ \lambda_1 &\sim \mu_2^c \lambda_2^d, \end{aligned} \tag{5.2}$$

with $ad - bc = -1$, then $(Y_2)_{c/d}(K_2) \cong M_2(\lambda_1)$ has homology $\mathbb{Z}/4\mathbb{Z}$, so $|c| = 4$.

We can replace $Y_1$ with some $\frac{1}{k}$-surgery on $K_1$, and $K_1$ with the core of this surgery; this preserves the longitude $\lambda_1$ and the complement $M_1$ but replaces $\mu_1$ with $\mu_1\lambda_1^k$, so in these new coordinates the gluing map is

$$\begin{aligned} \mu_1 &\sim \mu_2^{a+kc} \lambda_2^{b+kd} \\ \lambda_1 &\sim \mu_2^c \lambda_2^d. \end{aligned}$$

We know that $a$ must be odd since $ad - bc = -1$ and $c = \pm 4$, so $a \equiv \pm 1 \pmod 4$ and this means that we can choose $k$ so that $a + kc = \pm 1$. In other words, we can choose $\mu_1$ so that $a = \pm 1$ in (5.2). Having done this, we reverse the orientation of $K_2$ if needed to fix $a = 1$, so that we have an identification of the form $\mu_1 \sim \mu_2 \lambda_2^b$; and then we replace $Y_2$ and $K_2$ with $\frac{1}{b}$-surgery on $K_2$ and the core of this surgery.

At this point we have arranged for the gluing map (5.2) to have the form

$$\begin{aligned} \mu_1 &\sim \mu_2 \\ \lambda_1 &\sim \mu_2^c \lambda_2^d \end{aligned}$$

for some integers $c = \pm 4$ and $d$, and the condition $ad - bc = -1$ means that $d = -1$. If $c = 4$ then we can reverse the orientations of both $Y_1$ and $Y_2$ (and hence of $Y$ itself), while fixing the orientations of the $K_i$; this fixes $\mu_1$ and $\mu_2$ while replacing $\lambda_1$ and $\lambda_2$ with their inverses, so that $\lambda_1 \sim \mu_2^c \lambda_2^{-1}$ becomes $\lambda_1 \sim \mu_2^{-c} \lambda_2^{-1}$, and thus we are left with $c = -4$ instead. Now we have $\mu_1 \sim \mu_2$ and $\lambda_1 \sim \mu_2^{-4} \lambda_2^{-1}$ as in (5.1), and moreover

$$\begin{aligned} M_1(\lambda_2^{-1}) &\cong M_1(\mu_1^4 \lambda_1) \cong (Y_1)_4(K_1), \\ M_2(\lambda_1^{-1}) &\cong M_2(\mu_2^4 \lambda_2) \cong (Y_2)_4(K_2), \end{aligned}$$

exactly as claimed. □

We introduce a pair of involutions of the pillowcase $X(T^2)$, given in $(\alpha, \beta)$ coordinates by

$$\sigma(\alpha, \beta) = (\alpha, 2\pi - (4\alpha + \beta)), \tag{5.3}$$

$$\tau(\alpha, \beta) = (\pi - \alpha, 2\pi - \beta). \tag{5.4}$$

It is straightforward to check that these are indeed involutions, and that they commute since

$$\begin{aligned}\sigma(\tau(\alpha,\beta)) &= \sigma(\pi-\alpha, 2\pi-\beta) \\ &= (\pi-\alpha, 2\pi-(4(\pi-\alpha)+(2\pi-\beta))) \\ &\sim (\pi-\alpha, 2\pi-(2\pi-(4\alpha+\beta))) \\ &= \tau(\alpha, 2\pi-(4\alpha+\beta)) \\ &= \tau(\sigma(\alpha,\beta)).\end{aligned}$$

**Lemma 5.7.** *Let $K$ be a knot in a homology sphere $Y$. Then the pillowcase image $i^*X(Y,K)$ is fixed setwise by the involution $\tau$ of* (5.4).

*Proof.* Letting $E_K = Y \setminus N(K)$, we note that $H_1(E_K) \cong \mathbb{Z}$ is generated by the meridian $\mu$ of $K$, and that the longitude $\lambda$ is nullhomologous. Given a point $(\alpha,\beta) \in i^*X(Y,K)$, corresponding to a representation $\rho : \pi_1(E_K) \to \mathrm{SU}(2)$ with

$$\rho(\mu) = e^{i\alpha}, \qquad \rho(\lambda) = e^{i\beta},$$

we fix the central character $\chi : \pi_1(E_K) \twoheadrightarrow H_1(E_k) \to \{\pm 1\}$ with $\chi(\mu) = -1$ and $\chi(\lambda) = 1$, and consider the representation

$$\rho' = \chi \cdot \rho : \pi_1(E_K) \to \mathrm{SU}(2).$$

This satisfies $\rho'(\mu) = e^{i(\pi+\alpha)}$ and $\rho'(\lambda) = e^{i\beta}$, so its pillowcase image

$$i^*([\rho']) = (\pi+\alpha, \beta) \sim (\pi-\alpha, 2\pi-\beta) = \tau(\alpha,\beta)$$

belongs to $i^*X(K)$ as well. □

With Proposition 5.6 in mind, we now introduce the following refinement of Proposition 5.5.

**Lemma 5.8.** *Let $K \subset Y$ be a knot in a homology sphere, with irreducible, boundary-incompressible complement, and suppose that $Y_4(K)$ is a lens space. Then there is a topologically embedded, closed curve*

$$\gamma \subset i^*X(Y,K)$$

*that is homologically essential in the twice-punctured pillowcase*

$$P = X(T^2) \setminus \{(0,\pi), (\pi,\pi)\},$$

*and that contains the point $(\frac{\pi}{4}, 0)$. This curve $\gamma$ does not contain any other point $(\alpha,\beta)$ with $4\alpha+\beta \equiv 0 \pmod{\pi}$, except possibly for $(\frac{\pi}{2}, 0)$.*

*Proof.* We let $\gamma_0$ be the curve provided by Proposition 5.5, noting that this curve avoids both $(0,0)$ and $(\pi,0)$. Since $\gamma_0$ generates $H_1(P) \cong \mathbb{Z}$, it has intersection number $\pm 1$ with the curve

$$L_\pi = \{4\alpha+\beta \equiv \pi \pmod{2\pi}\} = \{(\alpha, \pi-4\alpha) \mid 0 < \alpha < \pi\}$$

that ends at the punctures $(0,\pi)$ and $(\pi,\pi)$, as shown in Figure 2.

According to Lemma 5.2, the curve $\gamma_0$ can only meet $L_\pi$ at those points of $L_\pi$ where $\beta \equiv 0 \pmod{2\pi}$, namely $(\frac{\pi}{4}, 0)$ and $(\frac{3\pi}{4}, 0)$. If $\gamma_0$ passes through either of these points then there is a neighborhood of that point where $\gamma_0$ coincides with the edge $\{\beta \equiv 0 \pmod{2\pi}\}$, by Proposition 5.3, and so $\gamma_0$ meets $L_\pi$ transversely there. This means that the points of

$$\gamma_0 \cap L_\pi \subset \left\{(\tfrac{\pi}{4}, 0), (\tfrac{3\pi}{4}, 0)\right\}$$

each contribute $\pm 1$ to the intersection number $\gamma_0 \cdot L_\pi = \pm 1$, and so exactly one of these must be a point of intersection. If the point is $(\frac{\pi}{4}, 0)$, then we take $\gamma = \gamma_0$ and we are done. If instead it is $(\frac{3\pi}{4}, 0)$, then we note that $i^*X(Y, K)$ is closed under the involution $\tau$ of (5.4), by Lemma 5.7, so we take $\gamma = \tau(\gamma_0)$ instead.

As for other points $(\alpha, \beta) \in \gamma$ with $4\alpha + \beta \equiv 0 \pmod{\pi}$, Lemma 5.2 says that these must have $\beta \equiv 0 \pmod{2\pi}$, and then we have $\alpha = \frac{k\pi}{4}$ for some integer $k$, with $0 \leq k \leq 4$ corresponding to $0 \leq \alpha \leq \pi$. We have already excluded $k = 0, 3, 4$ and arranged that $(\frac{\pi}{4}, 0) \in \gamma$, so this leaves $(\frac{\pi}{2}, 0)$ (corresponding to $k = 2$) as the only remaining possibility. □

5.3. **Intersecting curves of characters.** The curves $\gamma$ produced by Lemma 5.8 will be key to finding representations of closed 3-manifold groups. We will repeatedly use the following two facts in our arguments:

- any pair of closed curves in the pillowcase has intersection number zero;
- and if $\gamma$ is an essential, simple closed curve in the twice-punctured pillowcase $P$, then it generates $H_1(P) \cong \mathbb{Z}$ and thus has intersection number $\pm 1$ with any arc connecting the two punctures.

The second of these facts already played a role in the proof of Lemma 5.8. We recall below that $\sigma$ denotes the involution (5.3) of the pillowcase.

**Lemma 5.9.** *Let $K_1 \subset Y_1$ and $K_2 \subset Y_2$ be knots in homology spheres. Form a closed 3-manifold $Y$ by gluing together their exteriors $M_i = Y_i \setminus N(K_i)$ as in* (5.1), *so that*

$$\mu_1 \sim \mu_2, \qquad \lambda_1^{-1} \sim \mu_2^4 \lambda_2.$$

*If the intersection*

$$i^*X(Y_1, K_1) \cap \sigma\big(i^*X(Y_2, K_2)\big)$$

*contains a point other than* $(0, 0)$, $(\frac{\pi}{2}, 0)$, *and* $(\pi, 0)$, *then there is a representation*

$$\pi_1(Y) \to \mathrm{SU}(2)$$

*with non-abelian image.*

*Proof.* Let $(\alpha, \beta)$ be a point of the intersection. Then there are representations

$$\rho_i : \pi_1(M_i) \to \mathrm{SU}(2)$$

with $\rho_1(\mu_1) = e^{i\alpha}$ and $\rho_1(\lambda_1) = e^{i\beta}$, and with

$$\rho_1(\mu_1) = e^{i\alpha} \qquad \rho_2(\mu_2) = e^{i\alpha_2}$$
$$\rho_1(\lambda_1) = e^{i\beta}, \qquad \rho_2(\lambda_2) = e^{i\beta_2}$$

such that $(\alpha, \beta) = \sigma(\alpha_2, \beta_2) = (\alpha_2, 2\pi - (4\alpha_2 + \beta_2))$. In particular we have

$$\rho_2(\mu_2) = e^{i\alpha_2} = e^{i\alpha} = \rho_1(\mu_1),$$
$$\rho_2(\mu_2^4\lambda_2) = e^{i(4\alpha_2+\beta_2)} = e^{i(2\pi-\beta)} = \rho_1(\lambda_1^{-1})$$

and therefore $\rho_1$ and $\rho_2$ glue together to give a representation $\rho : \pi_1(Y) \to \mathrm{SU}(2)$ whose image contains the images of both $\rho_1$ and $\rho_2$.

Now if $\beta \not\equiv 0 \pmod{2\pi}$ then $\rho_1$ has non-abelian image, and likewise if $\beta_2 \not\equiv 0 \pmod{2\pi}$ then $\rho_2$ has non-abelian image. Thus $\rho$ is non-abelian unless both $\beta$ and $\beta_2$ are multiples of

$2\pi$. Since $\sigma$ is an involution we have $(\alpha_2, \beta_2) = \sigma(\alpha, \beta) = (\alpha, 2\pi - (4\alpha + \beta))$, so $\beta \equiv \beta_2 \equiv 0 \pmod{2\pi}$ is equivalent to

$$\beta \equiv 4\alpha + \beta \equiv 0 \pmod{2\pi}$$

and this corresponds to the three points $(0, 0)$, $(\frac{\pi}{2}, 0)$, $(\pi, 0)$ in the pillowcase. Thus any intersection point away from these three gives rise to a non-abelian $\rho$, as desired. □

From now on we will repeatedly use the following hypotheses.

**Setup 5.10.** Let $K_1 \subset Y_1$ and $K_2 \subset Y_2$ be knots in homology spheres, with the properties that each exterior $M_j = Y_j \setminus N(K_j)$ is irreducible and boundary-incompressible, and that each 4-surgery

$$(Y_j)_4(K_j), \qquad j = 1, 2$$

is a lens space. Form a closed 3-manifold $Y$ by gluing $M_1$ to $M_2$ by the map

$$\mu_1 \sim \mu_2,$$
$$\lambda_1^{-1} \sim \mu_2^4 \lambda_2$$

as in (5.1). Finally, let $\gamma_j \subset i^* X(Y_j, K_j)$ be the embedded closed curves in the pillowcase provided by Lemma 5.8, each of which avoids the lines $\{4\alpha + \beta \in \pi\mathbb{Z}\}$ except at $(\frac{\pi}{4}, 0)$ and possibly $(\frac{\pi}{2}, 0)$.

**Lemma 5.11.** *Assume Setup 5.10, and let*

$$c_j \subset i^* X(Y_j, K_j), \qquad j = 1, 2$$

*be closed, embedded curves that avoid the points* $(0, 0)$ *and* $(\pi, 0)$*, such as* $\gamma_j$ *or* $\tau(\gamma_j)$*. If the intersection*

$$c_1 \cap \sigma(c_2)$$

*is nonempty, then there is a non-abelian representation* $\rho : \pi_1(Y) \to \mathrm{SU}(2)$*.*

*Proof.* By Lemma 5.9 it suffices to show that $c_1$ and $\sigma(c_2)$ intersect in a point other than $(0, 0)$, $(\frac{\pi}{2}, 0)$, or $(\pi, 0)$. By hypothesis they avoid the first and last of these, so we need only consider the case where $c_1$ and $\sigma(c_2)$ both pass through $(\frac{\pi}{2}, 0)$. Since $(\frac{\pi}{2}, 0)$ is fixed by $\sigma$, this means that both of the $c_i$ contain $(\frac{\pi}{2}, 0)$.

According to Proposition 5.3, there is a neighborhood of $(\frac{\pi}{2}, 0)$ in the pillowcase on which each $i^* X(Y_j, K_j)$ coincides with the line $\beta \equiv 0 \pmod{2\pi}$. This means that on a sufficiently small neighborhood $U$ of $(\frac{\pi}{2}, 0)$, we have

$$c_1 \cap U = U \cap \{\beta \equiv 0 \pmod{2\pi}\}$$
$$\sigma(c_2) \cap U = U \cap \{4\alpha + \beta \equiv 0 \pmod{2\pi}\},$$

and so $c_1$ meets $\sigma(c_2)$ transversely at $(\frac{\pi}{2}, 0)$. In particular, the point $(\frac{\pi}{2}, 0)$ contributes $\pm 1$ to the intersection number

$$c_1 \cdot \sigma(c_2) = 0,$$

so there must be at least one other point of intersection $(\alpha, \beta) \in c_1 \cap \sigma(c_2)$ and this provides the desired $\rho$. □

In what follows, we will repeatedly refer to the pair of arcs

$$L_\theta = \{(\alpha, \beta) \mid 4\alpha + \beta \equiv \theta \pmod{2\pi}\} \qquad (\theta = 0, \pi) \tag{5.5}$$

in the pillowcase.

**Proposition 5.12.** *Assume Setup 5.10, and suppose further that neither $\gamma_1$ nor $\gamma_2$ contains the point $(\frac{\pi}{2}, 0)$. Then there is a non-abelian representation $\rho : \pi_1(Y) \to \mathrm{SU}(2)$.*

*Proof.* We recall from Lemma 5.7 that $\tau(\gamma_1) \subset i^*X(Y_1, K_1)$; moreover, the points $(0, 0)$ and $(\pi, 0)$ in the pillowcase are fixed by $\tau$, so if $\gamma_1$ avoids them both then so does $\tau(\gamma_1)$. Thus by Lemma 5.11, it suffices to show that either $\gamma_1$ or $\tau(\gamma_1)$ intersects $\sigma(\gamma_2)$. We will therefore suppose for the sake of a contradiction that

$$\big(\gamma_1 \cup \tau(\gamma_1)\big) \cap \sigma(\gamma_2) = \emptyset.$$

We first record that according to Setup 5.10, the curves $\gamma_1$ and $\gamma_2$ can only possibly meet the line $L_0$ at $(\frac{\pi}{2}, 0)$. By assumption they both avoid this point, so in fact

$$\gamma_1 \cap L_0 = \gamma_2 \cap L_0 = \emptyset.$$

Applying $\tau$ to $\gamma_1 \cap L_0$ and observing that $\tau(L_0) = L_0$ tells us that

$$\tau(\gamma_1) \cap L_0 = \emptyset$$

as well, while if we apply $\sigma$ to $\gamma_2 \cap L_0$ then we see that

$$\sigma(\gamma_2) \cap \{\beta \equiv 0 \pmod{2\pi}\} = \emptyset$$

since $\sigma$ sends $L_0$ to the line $\beta \equiv 0$.

By Proposition 5.3, the curve $\gamma_1$ intersects $L_\pi$ transversely at their sole point $(\frac{\pi}{4}, 0)$ of intersection, and similarly $\tau(\gamma_1)$ meets $L_\pi$ transversely at $(\frac{3\pi}{4}, 0)$ and nowhere else. Thus $\gamma_1$ and $\tau(\gamma_1)$ separate $L_\pi$ into three segments, which we label

$$\begin{aligned} L^\ell_\pi &= \{(\alpha, \pi - 4\alpha) \mid 0 \le \alpha < \tfrac{\pi}{4}\}, \\ L^m_\pi &= \{(\alpha, 3\pi - 4\alpha) \mid \tfrac{\pi}{4} < \alpha < \tfrac{3\pi}{4}\}, \\ L^r_\pi &= \{(\alpha, 5\pi - 4\alpha) \mid \tfrac{3\pi}{4} < \alpha \le \pi\}. \end{aligned}$$

The union $\gamma_1 \cup \tau(\gamma_1)$ splits the pillowcase into various path components, and we let

$$X_\ell,\ X_m,\ X_r \subset X(T^2) \setminus \big(\gamma_1 \cup \tau(\gamma_1)\big)$$

denote the path components containing $L^\ell_\pi$, $L^m_\pi$, and $L^r_\pi$ respectively. Since $L^\ell_\pi$ lies in a different component of $X(T^2) \setminus \gamma_1$ than $L^m_\pi$ and $L^r_\pi$, and similarly $L^r_\pi$ lies in a different component of $X(T^2) \setminus \tau(\gamma_1)$ than $L^\ell_\pi$ and $L^m_\pi$, it follows that the path components $X_\ell$, $X_m$, and $X_r$ are all distinct.

Next, we observe that the curves $\sigma(\gamma_2)$ and $L_0$ are by assumption disjoint from $\gamma_1 \cup \tau(\gamma_1)$, so each one lies entirely within some path component of the complement. Since $\sigma$ restricts to an involution of the twice-punctured pillowcase, the curve $\sigma(\gamma_2)$ generates the homology of the latter just as $\gamma_2$ does; meanwhile $L_\pi$ is an arc with endpoints at the two punctures $(0, \pi)$ and $(\pi, \pi)$, so we must have

$$\sigma(\gamma_2) \cap L_\pi \neq \emptyset.$$

At the same time $\sigma(\gamma_2)$ also contains the point $\sigma(\frac{\pi}{4}, 0) = (\frac{\pi}{4}, \pi) \in L_0$, so it contains a path with one endpoint on $L_0$ and the other endpoint on $L_\pi$. The endpoint of this path on $L_\pi$ lies in one of the segments $L^\ell_\pi$, $L^m_\pi$, or $L^r_\pi$, since the remaining points $(\frac{\pi}{4}, 0)$ and $(\frac{3\pi}{4}, 0)$ of $L_\pi$ lie on $\gamma_1$ and $\tau(\gamma_1)$ respectively, so $L_0$ is in the same path component as that segment. Thus exactly one of

$$L_0 \subset X_\ell \quad \text{or} \quad L_0 \subset X_m \quad \text{or} \quad L_0 \subset X_r$$

must be true.

Since the set $\gamma_1 \cup \tau(\gamma_1)$ is $\tau$-invariant, the involution $\tau$ permutes the path components of their complement, and it moreover fixes the path component containing $L_0$ since $\tau(L_0) = L_0$. We observe $\tau$ exchanges the points

$$(0, \pi) \in L^\ell_\pi \subset X_\ell \qquad \text{and} \qquad (\pi, \pi) \in L^r_\pi \subset X_r,$$

but it fixes the point $(\frac{\pi}{2}, \pi) \in L^m_\pi$, so $\tau$ exchanges $X_\ell$ and $X_r$ while fixing $X_m$ and therefore

$$L_0 \subset X_m.$$

Finally, we build another path from $(0, \pi)$ to $(\pi, \pi)$ as the union

$$L^\ell_\pi \cup \big([\tfrac{\pi}{4}, \tfrac{3\pi}{4}] \times \{0\}\big) \cup L^r_\pi.$$

Since $\sigma(\gamma_2)$ is homologically essential in the twice-punctured pillowcase, it must intersect this path somewhere. But we saw that $\sigma(\gamma_2)$ is disjoint from the middle segment, since in fact it avoids the entire line $\{\beta \equiv 0 \pmod{2\pi}\}$, so it follows that either

$$\sigma(\gamma_2) \cap L^\ell_\pi \neq \emptyset \quad \text{or} \quad \sigma(\gamma_2) \cap L^r_\pi \neq \emptyset.$$

This means that $\sigma(\gamma_2)$ intersects at least one of the path components $X_\ell$ and $X_r$, and at the same time we have also seen that it contains the point

$$(\tfrac{\pi}{4}, \pi) \in L_0 \subset X_m.$$

But then the curve $\sigma(\gamma_2)$ contains a path from $X_m$ to either $X_\ell$ or $X_r$, contradicting the fact that each of these path components are distinct. We conclude that $\sigma(\gamma_2)$ must intersect either $\gamma_1$ or $\tau(\gamma_1)$ after all, and this provides the desired representation $\rho$. □

5.4. **Instanton knot homology and** $\mathrm{SU}(2)$ **representations.** In this subsection we suppose that $Y = M_1 \cup_{T^2} M_2$ is formed as in Setup 5.10, and that $Y$ is SU(2)-abelian. We will use the curve of characters $\sigma(\gamma_2)$ for $K_2$ in the pillowcase, and specifically the fact that it mostly avoids the pillowcase image $i^* X(Y_1, K_1)$, to conclude that the *instanton knot homology*

$$KHI(Y_1, K_1)$$

defined by Kronheimer and Mrowka [KM10] must be small. The following lower bound on the rank of *KHI* will then give us a contradiction, from which we can conclude that $Y$ must not be SU(2)-abelian after all.

**Lemma 5.13.** *Let $K \subset Y$ be a knot in a homology sphere with irreducible, boundary-incompressible complement. Then* $\dim KHI(Y, K) \geq 2$*.*

*Proof.* Kronheimer and Mrowka [KM10] define instanton knot homology as the sutured instanton homology

$$KHI(Y, K) = SHI(Y(K)),$$

where the sutured manifold $Y(K)$ is the complement $M = Y \setminus N(K)$ with a pair of oppositely oriented meridional sutures. Since $M = Y \setminus N(K)$ is irreducible, the sutured manifold $Y(K)$ is taut, and in particular its sutured instanton homology is nonzero [KM10, Theorem 7.12].

Now we know that $H_2(M) = 0$ and that $SHI(Y(K))$ is nonzero, so a theorem of Ghosh and Li [GL23, Theorem 1.2] tells us that

$$\dim SHI(Y(K)) < 2$$

if and only if $Y(K)$ is a product sutured manifold, in which case

$$Y(K) \cong (\Sigma \times [-1,1], \partial\Sigma \times \{0\})$$

for some compact surface $\Sigma$ with boundary. (This theorem is in turn a generalization of [KM10, Theorem 7.18], which required the additional hypothesis that $Y(K)$ is a homology product.) But since the positive and negative regions $R_\pm \subset \partial M$ are annuli, this could only be possible if $\Sigma$ were an annulus, in which case $M \cong \Sigma \times [-1,1]$ would be a solid torus. Since $\partial M$ is incompressible, we conclude that $\dim KHI(Y,K) = \dim SHI(Y(K))$ is at least 2 after all. □

Our goal will be to use $\sigma(\gamma_2)$ to construct a curve $\vec{c}' \subset X(T^2)$ and isotopy $h_t$ satisfying the hypotheses of the following theorem, and thus bound $\dim KHI(Y_1, K_1)$ from above.

**Theorem 5.14** ([SZ22b, Theorem 4.8])**.** *Let $K \subset Y$ be a knot in a homology 3-sphere, and suppose we have a smooth, simple closed curve $\vec{c}' \subset X(T^2)$ and an area-preserving isotopy*

$$h_t : X(T^2) \to X(T^2), \qquad 0 \le t \le 1$$

*that takes $\vec{c}'$ to the line $\{\alpha = \frac{\pi}{2}\}$ and fixes the four points $(0,0)$, $(0,\pi)$, $(\pi,0)$, and $(\pi,\pi)$. Suppose moreover that*

*(1) $\vec{c}'$ is disjoint from $i^*X^{irr}(Y,K)$,*
*(2) and $\vec{c}'$ intersects the line*

$$\{\beta \equiv 0 \pmod{2\pi}\} \subset X(T^2)$$

*transversely in $n$ points $(\alpha_1, 0), \ldots, (\alpha_n, 0)$, with $\Delta_K(e^{2i\alpha_j}) \neq 0$ for all $j$.*

*Then $\dim KHI(K) \le n$.*

**Remark 5.15.** The statement of Theorem 5.14 in [SZ22b] assumes that $Y \cong S^3$, but the proof applies verbatim when $Y$ is an arbitrary homology sphere.

With these prerequisites at hand, we now devote the remainder of this subsection to the proof of the following proposition.

**Proposition 5.16.** *Assume Setup 5.10, and suppose that $(\frac{\pi}{2}, 0)$ lies on at least one of the curves $\gamma_1$ and $\gamma_2$. Then there is a representation $\rho : \pi_1(Y) \to \mathrm{SU}(2)$ with non-abelian image.*

We begin with a special case of the proposition that will simplify our subsequent application of Theorem 5.14 in the general case.

**Lemma 5.17.** *Assume Setup 5.10, and suppose in addition that $(\frac{\pi}{2}, 0) \in \gamma_2$. If $\sigma(\gamma_2)$ intersects either of the segments*

$$L^\ell_\pi = \{(\alpha, \pi - 4\alpha) \mid 0 \le \alpha < \tfrac{\pi}{4}\}$$
$$L^r_\pi = \{(\alpha, 5\pi - 4\alpha) \mid \tfrac{3\pi}{4} < \alpha \le \pi\} \tag{5.6}$$

*of the line $L_\pi = \{4\alpha + \beta \equiv \pi \pmod{2\pi}\}$, then there is a representation $\pi_1(Y) \to \mathrm{SU}(2)$ with non-abelian image.*

*Proof.* By hypothesis the image $\sigma(\gamma_2)$ contains the point $\sigma(\frac{\pi}{2}, 0) = (\frac{\pi}{2}, 0)$. Supposing for now that $\sigma(\gamma_2)$ intersects $L^\ell_\pi$, say at some point $p$, then the union

$$L^\ell_\pi \cup \sigma(\gamma_2)$$

contains a path $\phi$ from $(0,\pi)$ to $(\frac{\pi}{2},0)$, realized by following $L^\ell_\pi$ from $(0,\pi)$ to $p$ and then traveling from $p$ to $(\frac{\pi}{2},0)$ along $\sigma(\gamma_2)$. Now $\tau(\phi)$ is (up to reversing the direction of travel) a path from $\tau(\frac{\pi}{2},0)=(\frac{\pi}{2},0)$ to $\tau(0,\pi)=(\pi,\pi)$, and it is contained in

$$\tau(L^\ell_\pi)\cup\tau\big(\sigma(\gamma_2)\big)=L^r_\pi\cup\sigma\big(\tau(\gamma_2)\big),$$

so we concatenate $\phi$ and $\tau(\phi)$ to get a path $\bar\phi$ inside

$$L^\ell_\pi\cup\sigma(\gamma_2)\cup\sigma\big(\tau(\gamma_2)\big)\cup L^r_\pi \tag{5.7}$$

from $(0,\pi)$ to $(\pi,\pi)$.

Since $\gamma_1$ is homologically essential in the twice-punctured pillowcase, the path $\bar\phi$ from one puncture to the other must intersect it somewhere. Lemma 5.2 guarantees that $\gamma_1$ is disjoint from both $L^\ell_\pi$ and $L^r_\pi$, so we must have either

$$\gamma_1\cap\sigma(\gamma_2)\neq\emptyset \quad\text{or}\quad \gamma_1\cap\sigma\big(\tau(\gamma_2)\big)\neq\emptyset,$$

and in either case Lemma 5.11 provides the desired representation $\pi_1(Y)\to \mathrm{SU}(2)$.

The case where $\sigma(\gamma_2)$ intersects $L^r_\pi$ is nearly identical, except that this leads to a path $\phi$ from $(\frac{\pi}{2},0)$ to $(\pi,\pi)$ inside $\sigma(\gamma_2)\cup L^r_\pi$. In this case the union $\phi\cup\tau(\phi)$ still gives us a path from $(0,\pi)$ to $(\pi,\pi)$ inside (5.7) and we proceed exactly as before. □

We are looking for a non-abelian representation $\rho:\pi_1(Y)\to\mathrm{SU}(2)$, and if neither curve $\gamma_j$ passes through $(\frac{\pi}{2},0)$ then Proposition 5.12 provides such a $\rho$, so we will assume that at least one $\gamma_j$ does contain this point. Since the roles of the two $(Y_j,K_j)$ are completely interchangeable, we will assume without loss of generality that

$$\gamma_2\cap L_0=\big\{(\tfrac{\pi}{2},0)\big\}.$$

With $L^\ell_\pi$ and $L^r_\pi$ defined as in (5.6), we will also assume that

$$\sigma(\gamma_2)\cap\Big(L^\ell_\pi\cup L^r_\pi\Big)=\emptyset, \tag{5.8}$$

because otherwise Lemma 5.17 provides a non-abelian representation $\pi_1(Y)\to\mathrm{SU}(2)$.

In addition to the above identification of $\gamma_2\cap L_0$, we also know from Setup 5.10 that $\gamma_2\cap L_\pi=\{(\frac{\pi}{4},0)\}$. These claims about each $\gamma_2\cap L_\theta$ are equivalent to

$$\begin{aligned}\sigma(\gamma_2)\cap\{(\alpha,\beta)\mid\beta\equiv0\pmod{2\pi}\}&=\big\{(\tfrac{\pi}{2},0)\big\},\\ \sigma(\gamma_2)\cap\{(\alpha,\beta)\mid\beta\equiv\pi\pmod{2\pi}\}&=\big\{(\tfrac{\pi}{4},\pi)\big\}.\end{aligned} \tag{5.9}$$

Moreover, since 4-surgery on $K_2\subset Y_2$ is a lens space, Proposition 5.3 says that $\gamma_2$ coincides with the line $\{\beta\equiv0\pmod{2\pi}\}$ on some neighborhood of $(\frac{\pi}{4},0)$ and of $(\frac{\pi}{2},0)$. Applying $\sigma$, we see that there are likewise open neighborhoods of $(\frac{\pi}{4},\pi)$ and of $(\frac{\pi}{2},0)$ on which $\sigma(\gamma_2)$ coincides with the line $L_0=\{4\alpha+\beta\equiv0\pmod{2\pi}\}$. In particular, the intersections (5.9) are both transverse.

The line segments $L^\ell_\pi$, $L^r_\pi$, and $\{\beta\equiv0,\pi\pmod{2\pi}\}$ collectively divide the pillowcase into a pair of closed disks of equal area: we write

$$X(T^2)=D_{\mathrm{top}}\cup D_{\mathrm{bot}},$$

where $D_{\mathrm{top}}$ and $D_{\mathrm{bot}}$ are the regions containing $(\frac{\pi}{2},\frac{3\pi}{2})$ and $(\frac{\pi}{2},\frac{\pi}{2})$ respectively, so that $\tau(D_{\mathrm{top}})=D_{\mathrm{bot}}$. In both sides of Figure 3 we have shaded the region $D_{\mathrm{top}}$. We note that

$$\sigma(\gamma_2)\cap\partial D_{\mathrm{top}}=\big\{(\tfrac{\pi}{2},0),\ (\tfrac{\pi}{4},\pi)\big\} \tag{5.10}$$

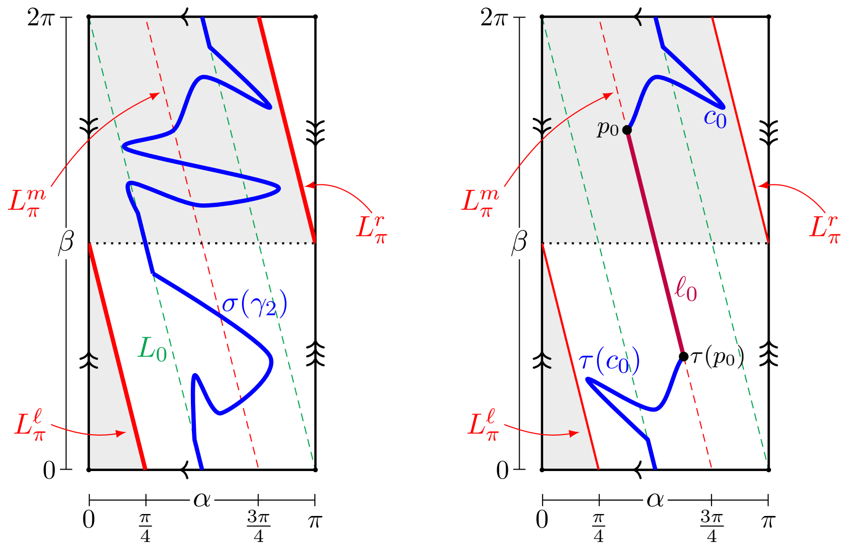


FIGURE 3. Left: the curve $\sigma(\gamma_2)$ avoids the line segments $L_\pi^\ell$ and $L_\pi^r$ and crosses $\beta \in \pi\mathbb{Z}$ only at $(\frac{\pi}{2}, 0)$ and $(\frac{\pi}{4}, \pi)$, as promised in (5.10). Right: we build a $\tau$-invariant, essential closed curve $\bar{c}$ in the pillowcase out of the arc $c_0$ from $(\frac{\pi}{2}, 2\pi)$ to $L_\pi^m$, a portion $\ell_0$ of $L_\pi^m$, and the image $\tau(c_0)$. In both pictures we have shaded the disk $D_{\text{top}}$ for clarity.

by (5.8) and (5.9), and that this intersection is transverse.

We now observe that by (5.10), the intersection

$$\sigma(\gamma_2) \cap D_{\text{top}}$$

defines a path from $(\frac{\pi}{2}, 2\pi) = (\frac{\pi}{2}, 0)$ to $(\frac{\pi}{4}, \pi)$ that avoids $\partial D_{\text{top}}$ except at its endpoints. This path must then cross the segment

$$L_\pi^m = \{(\alpha, 3\pi - 4\alpha) \mid \tfrac{\pi}{4} < \alpha < \tfrac{3\pi}{4}\}$$

of $L_\pi$, because $L_\pi^m$ separates $(\frac{\pi}{2}, 2\pi)$ from $(\frac{\pi}{4}, \pi)$ in the disk $D_{\text{top}}$. See the left side of Figure 3.

With this in mind, we let

$$c_0 \subset \sigma(\gamma_2) \cap D_{\text{top}}$$

be the portion of this path (including both endpoints) from $(\frac{\pi}{2}, 2\pi)$ to the first point

$$p_0 = (\alpha_0, \beta_0) \in L_\pi^m$$

where this path meets $L_\pi^m$; we note that $\pi < \beta_0 < 2\pi$ and thus $\frac{\pi}{4} < \alpha_0 < \frac{\pi}{2}$. We then take the line segment

$$\ell_0 = \{(\alpha, 3\pi - 4\alpha) \mid \alpha_0 \leq \alpha \leq \pi - \alpha_0\} \subset L_\pi^m \tag{5.11}$$

from $p_0$ to $\tau(p_0) = (\pi - \alpha_0, 2\pi - \beta_0)$, and define the simple closed curve

$$\bar{c} = c_0 \cup \ell_0 \cup \tau(c_0)$$

in the pillowcase, as on the right side of Figure 3. This is indeed a simple curve because by construction its three sections $c_0$, $\ell_0$, and $\tau(c_0)$ intersect each other only at their respective endpoints, which are among the points $(\frac{\pi}{2},0)$, $p_0$, and $\tau(p_0)$. Moreover, since $\tau(\ell_0)=\ell_0$ it follows that $\tau(\bar{c})=\bar{c}$.

**Lemma 5.18.** *Suppose that there are no non-abelian representations $\pi_1(Y)\to\mathrm{SU}(2)$. Then there is an open neighborhood $U$ of $\bar{c}\subset X(T^2)$ with the property that the intersection*

$$i^*X^{irr}(Y_1,K_1)\cap U$$

*is empty.*

*Proof.* Proposition 5.3 says that there is an open neighborhood $V\subset X(T^2)$ of the lines

$$L_0\cup L_\pi=\{(\alpha,\beta)\mid 4\alpha+\beta\in\pi\mathbb{Z}\}$$

that is disjoint from $i^*X^{\mathrm{irr}}(Y_1,K_1)$. Moreover, since $Y$ is SU(2)-abelian, Lemma 5.9 says that the entire pillowcase image $i^*X(Y_1,K_1)$ does not meet the subset

$$c_0\cup\tau(c_0)\subset\sigma(\gamma_2)\cup\tau\big(\sigma(\gamma_2)\big)=\sigma(\gamma_2)\cup\sigma\big(\tau(\gamma_2)\big)$$

of the pillowcase, except possibly at some of the points

$$(0,0),\ (\tfrac{\pi}{2},0),\ (\pi,0)\in L_0\subset V.$$

We let $V_0\subset V$ be an open neighborhood of these three points, and then we have

$$i^*X^{\mathrm{irr}}(Y_1,K_1)\subset i^*X(Y_1,K_1)\setminus V_0.$$

The set on the right is closed and disjoint from the closed set $c_0\cup\tau(c_0)$, so it is disjoint from an entire open neighborhood $W$ of $c_0\cup\tau(c_0)$, and then

$$i^*X^{\mathrm{irr}}(Y_1,K_1)\cap W=\emptyset.$$

We now let $U=V\cup W$, which contains all of $\bar{c}$ since $\ell_0\subset V$ and $c_0\cup\tau(c_0)\subset W$, and we observe that $U$ is disjoint from $i^*X^{\mathrm{irr}}(Y_1,K_1)$ since both $V$ and $W$ are. ☐

The curve $\bar{c}$ may not be smooth in general, but there is some small $\epsilon>0$ such that it coincides with $L_0$ and with $L_\pi$ on $2\epsilon$-neighborhoods of $(\frac{\pi}{2},0)$ and of $(\frac{\pi}{2},\pi)$, respectively. Given the neighborhood $U$ of Lemma 5.18, we can thus take a $C^0$-close approximation

$$\bar{c}'\subset U\subset X(T^2)\setminus i^*X^{\mathrm{irr}}(Y_1,K_1) \tag{5.12}$$

of $\bar{c}$ such that

(1) $\bar{c}'$ is smooth and $\tau$-invariant,
(2) and $\bar{c}'$ coincides with $\bar{c}$ in $\epsilon$-neighborhoods of $(\frac{\pi}{2},0)$ and $(\frac{\pi}{2},\pi)$, i.e.,

$$\bar{c}'\cap D_\epsilon(\tfrac{\pi}{2},0)=L_0\cap D_\epsilon(\tfrac{\pi}{2},0),$$
$$\bar{c}'\cap D_\epsilon(\tfrac{\pi}{2},\pi)=L_\pi\cap D_\epsilon(\tfrac{\pi}{2},\pi).$$

In particular $\bar{c}'$ meets $\{\beta\equiv 0\pmod{2\pi}\}$ transversely at $(\frac{\pi}{2},0)$, just as $\bar{c}$ does.

We achieve the $\tau$-invariance by first constructing the arc $\bar{c}'\cap D_{\mathrm{top}}$ and then using $\tau$ to extend it to $D_{\mathrm{bot}}$; if the initial arc is sufficiently $C^0$-close to $\bar{c}$ then both it and its image under $\tau$ will lie in $U$.

**Lemma 5.19.** *There is an area-preserving isotopy of the pillowcase that fixes the corners $(0,0)$, $(0,\pi)$, $(\pi,0)$, and $(\pi,\pi)$, and that takes the curve $\bar{c}'$ of (5.12) to the line $\{\alpha=\frac{\pi}{2}\}$.*

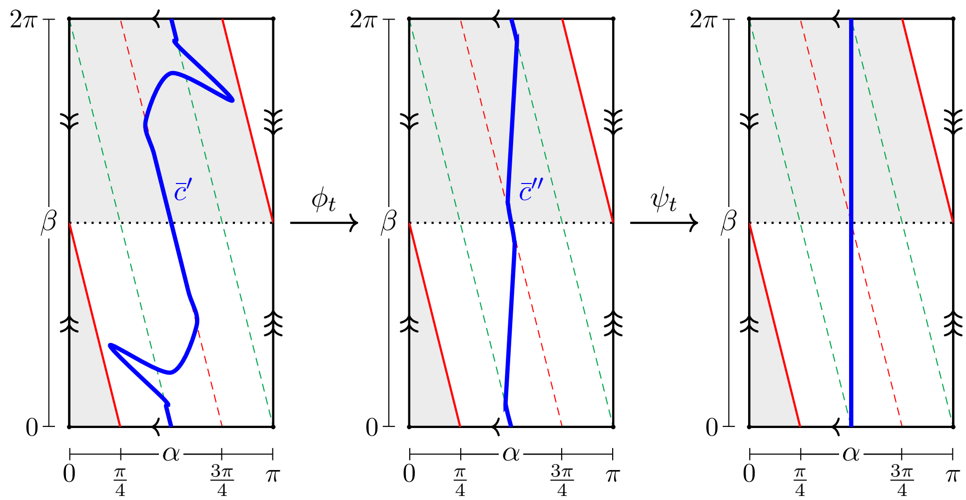


FIGURE 4. A $\tau$-equivariant isotopy carrying $\vec{c}'$ to the straight line $\{\alpha = \frac{\pi}{2}\}$. In each picture the disk $D_{\text{top}}$ is shaded.

*Proof.* By taking a smaller value of $\epsilon > 0$ as needed, we can assume that $\vec{c}'$ only intersects an $\epsilon$-neighborhood of $\partial D_{\text{top}}$ near $(\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \pi)$, where it coincides with $L_0$ and $L_\pi$ respectively.

We build a smooth curve $\vec{c}''$ isotopic to $\vec{c}'$ by taking a piecewise linear path in $D_{\text{top}}$ from

$$(\tfrac{\pi}{2}, 2\pi) \quad \text{to} \quad (\tfrac{\pi}{2} + \tfrac{\epsilon}{2}, 2\pi - 2\epsilon) \quad \text{to} \quad (\tfrac{\pi}{2} - \tfrac{\epsilon}{2}, \tfrac{\pi}{2} + 2\epsilon) \quad \text{to} \quad (\tfrac{\pi}{2}, \pi),$$

rounding corners in an $\frac{\epsilon}{2}$-neighborhood of the middle two vertices, and then using $\tau$ to extend this to $D_{\text{bot}}$. Then $\vec{c}''$ agrees with $\vec{c}'$ in an $\epsilon$-neighborhood of $\partial D_{\text{top}}$, and we can arrange the corner-rounding process so that $\vec{c}''$ intersects every horizontal line of the form $\{\beta = \beta_0\}$, $\beta_0 \in \mathbb{R}/2\pi\mathbb{Z}$, in a single point. See the middle of Figure 4.

We now choose a smooth isotopy of the disk $D_{\text{top}}$ that fixes the above $\epsilon$-neighborhood of $\partial D_{\text{top}}$ and takes the arc $\vec{c}' \cap D_{\text{top}}$ to $\vec{c}'' \cap D_{\text{top}}$. We extend this $\tau$-equivariantly across $D_{\text{bot}}$ to get an isotopy

$$\phi_t : X(T^2) \to X(T^2), \qquad t \in [0, 1]$$

that satisfies $\phi_0 = \text{id}$ and $\phi_1(\vec{c}') = \vec{c}''$, and that fixes the corners $(0, 0)$, $(0, \pi)$, $(\pi, 0)$, and $(\pi, \pi)$. We then apply another $\tau$-invariant isotopy of the form

$$\psi_t(\alpha, \beta) = (f_\beta(\alpha), \beta), \qquad t \in [0, 1],$$

supported in the region

$$\frac{\pi}{2} - \epsilon < \alpha < \frac{\pi}{2} + \epsilon$$

of the pillowcase, that takes $\phi_1(\vec{c}') = \vec{c}''$ to the line $\{\alpha = \frac{\pi}{2}\}$. Each of these is illustrated in Figure 4.

The composition of $\phi$ and $\psi$ is an isotopy of the pillowcase that fixes each of the four corners, and that carries $\vec{c}'$ to $\{\alpha = \frac{\pi}{2}\}$ through $\tau$-invariant curves $\vec{c}'_t$. These curves separate $X(T^2)$ into components that are exchanged by the isometry $\tau$ and must therefore have equal area. We can therefore use [SZ22b, Lemma 4.9] to promote $\phi * \psi$ to a smooth isotopy

$$h_t : X(T^2) \to X(T^2), \qquad t \in [0, 1],$$

fixing a neighborhood of the four corners for all $t$, such that each $h_t$ is a symplectomorphism satisfying $h_t(\vec{c}'_0) = \vec{c}'_t$. Then $h_t$ is the desired isotopy from our original curve $\vec{c}'_0 = \vec{c}'$ to the line $\vec{c}'_1 = \{\alpha = \frac{\pi}{2}\}$. □

*Proof of Proposition 5.16.* As above, we assume without loss of generality that $(\frac{\pi}{2}, 0) \in \gamma_2$. Assuming that the desired $\rho$ does not exist, Lemma 5.17 says that the curve $\sigma(\gamma_2)$ is disjoint from the line segments $L^\ell_\pi$ and $L^r_\pi$ of (5.6), so we have subsequently used $\sigma(\gamma_2)$ to construct

(1) A smooth simple closed curve $\vec{c}'$ in the pillowcase (as in (5.12)) such that
   (a) $\vec{c}'$ is disjoint from $i^*X^{\mathrm{irr}}(Y_1, K_1)$,
   (b) and the intersection
$$\vec{c}' \cap \{\beta \equiv 0 \pmod{2\pi}\} = \big\{(\tfrac{\pi}{2}, 0)\big\}$$
   is transverse.
(2) An area-preserving isotopy $h_t : X(T^2) \to X(T^2)$ that fixes the four corners of the pillowcase and sends $\vec{c}'$ to the line $\{\alpha = \frac{\pi}{2}\}$, by Lemma 5.19.

Since $\Delta_{K_1}(e^{2i\cdot\pi/2}) = \Delta_{K_1}(-1)$ is nonzero, we can now apply Theorem 5.14 to conclude that
$$\dim KHI(Y_1, K_1) \leq 1.$$
But this contradicts Lemma 5.13, which asserts that
$$\dim KHI(Y_1, K_1) \geq 2.$$
The representation $\rho$ must therefore exist after all. □

We can finally prove the main theorem of this section.

*Proof of Theorem 5.1.* Proposition 5.6 says that we can write $Y = M_1 \cup_{T^2} M_2$, where each $M_j$ is the exterior of a knot $K_j$ in a homology sphere $Y_j$ and the gluing maps satisfy
$$\mu_1 \sim \mu_2, \qquad\qquad \lambda_1^{-1} \sim \mu_2^4\lambda_2$$
as in (5.1). Then $(Y_1)_4(K_1) \cong M_1(\lambda_2)$ and $(Y_2)_4(K_2) \cong M_2(\lambda_1)$ are both lens spaces of order 4, so Lemma 5.8 provides us with simple closed curves $\gamma_j \subset i^*X(Y_j, K_j)$ for $j = 1, 2$, and all of the above is exactly as described in Setup 5.10.

Now if neither $\gamma_1$ nor $\gamma_2$ contains the point $(\frac{\pi}{2}, 0)$, then $\rho$ is provided by Proposition 5.12. Otherwise at least one of the $\gamma_j$ contains $(\frac{\pi}{2}, 0)$, and then Proposition 5.16 tells us that $\rho$ must exist. □

Since Theorem 5.1 has now been proved, this completes the proof of Theorem 1.4. □

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