Exercice 6

$$\begin{array}{ccc} S^1 & \to & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

$S^3 \subseteq \mathbb{C}^2$

$S^1 \subseteq \mathbb{C}$ unit cpx numbers

Here $S^1 \curvearrowright S^3$ is by isometries.

$\langle -,- \rangle_{\mathbb{C}^2}$ (sesquilinear) inner product

gives a Riem. metric

$\mathrm{Re}(\langle -,- \rangle_{\mathbb{C}})$ on $\mathbb{C}^2$,

which we restrict to S^3.

$$\mathrm{Re}\left(\left\langle \alpha \binom{z}{w}, \alpha \binom{z}{w} \right\rangle\right)$$

$$= \mathrm{Re}\left(|\alpha|^2 \left\langle \binom{z}{w}, \binom{z}{w} \right\rangle\right)$$

$\Longrightarrow$ get a connection by taking orthog. complement of VTP by Exercise 5.

$$TS^3_{(z,w)} = (z,w)^{\perp_{\mathbb{R}}}$$

for $(z,w) \in S^3$

<u>Pf</u>: γ a curve through (z,w)

$$0 = \frac{d}{dt}\Big|_{t=0} \underbrace{\langle \gamma(t), \gamma(t) \rangle_{\mathbb{C}}}_{\equiv 1}$$

$$= \langle \dot\gamma(0), \gamma(0) \rangle_{\mathbb{C}} + \langle \gamma(0), \dot\gamma(0) \rangle_{\mathbb{C}}$$

$$= 2\,\mathrm{Re}\,(\langle \dot\gamma(0), \underset{(z,w)}{\underset{\shortparallel}{\gamma(0)}} \rangle)$$

$\blacksquare$

$$VTS^3 = \langle i\,(z,w)\rangle_{\mathbb{R}} \quad \text{because}$$

$$\frac{d}{dt}\Big|_{t=0} (z,w)e^{it} = i\cdot(z,w)$$

$$\Rightarrow (VTS^3)^{\perp_{\mathbb{R}}} \text{ in } \mathbb{C}^2 \text{ is } i\cdot(z,w)^{\perp}$$

$$\text{Above} \Rightarrow (VTS^3)^{\perp_{\mathbb{R}}} \cap TS^3$$

$$= \langle i\,(z,w),\,(z,w)\rangle_{\mathbb{R}}^{\perp_{\mathbb{R}}}$$

$$= \langle (z,w)\rangle_{\mathbb{C}}^{\perp_{\mathbb{C}}}$$

real span

$$\overset{(*)}{=} \langle (\bar{w},-\bar{z}),\, i(\bar{w},-\bar{z})\rangle_{\mathbb{R}}$$

$$= \langle (\bar{w},-\bar{z})\rangle_{\mathbb{C}}$$

(*) because

$$\langle (\bar{w}, -\bar{z}), (z, w) \rangle$$

$$= wz - zw = 0$$

So $H_A = (VTS^3)^{\perp_{\mathbb{R}}} \cap TS^3$

$$= \left\langle \begin{pmatrix} \bar{w} \\ -\bar{z} \end{pmatrix}, i \begin{pmatrix} \bar{w} \\ -\bar{z} \end{pmatrix} \right\rangle_{\mathbb{R}}$$

What is the connection 1-form ω_A?

Needs to satisfy

$$\begin{cases} \omega_A(i(z,w)) \overset{!}{=} i \\ \text{and} \\ \omega_A \big|_{H_A} \overset{!}{\equiv} 0 \end{cases}$$

$$\omega_A = i\,\mathrm{Re}\left(\left\langle i\begin{pmatrix} z \\ w \end{pmatrix}, - \right\rangle_{\mathbb{C}}\right)$$

works:

- $$\omega_A\left(\begin{pmatrix} \bar{w} \\ -\bar{z} \end{pmatrix}\right) = i\,\mathrm{Re}\left(\left\langle i\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} \bar{w} \\ -\bar{z} \end{pmatrix}\right\rangle_{\mathbb{C}}\right)$$
$$= i\,\mathrm{Re}\left(-i\left(\bar{z}\bar{w} - \bar{w}\bar{z}\right)\right)$$
$$= 0$$

- Similar:
$$\omega_A\left(i\begin{pmatrix} \bar{w} \\ -\bar{z} \end{pmatrix}\right) = 0$$

- $$\omega_A\left(i\begin{pmatrix} z \\ w \end{pmatrix}\right)$$
$$= i\,\mathrm{Re}\left(\left\langle i\begin{pmatrix} z \\ w \end{pmatrix}, i\begin{pmatrix} z \\ w \end{pmatrix}\right\rangle\right)$$
$$= i\,\mathrm{Re}\left(\underbrace{\left(|z|^2 + |w|^2\right)}_{=1}\right)$$
$$= i.$$

In fact

$$\omega_A = i \cdot \mathrm{Im}\left(\left\langle \binom{z}{w}, - \right\rangle\right)$$

is a dif. expression for the same $i\mathbb{R}$-valued 1-form.

In diff-form notation:

$$\omega_A = i\,\mathrm{Im}\left(\bar{z}\,dz + \bar{w}\,dw\right)$$

(Rk:

$$\omega_A = \bar{z}\,dz + \bar{w}\,dw$$

if restricted to TS^3, but it's not an $i\mathbb{R}$-valued 1 form on $T\mathbb{C}^2$)

Exercise:

On $S^{2n+1} \subseteq \mathbb{C}^{n+1}$

$$\begin{array}{c} S^1 \longrightarrow S^{2n+1} =: P \\ \downarrow \\ \mathbb{CP}^n \end{array}$$

coo. $(z_0, \dots, z_n)$

$$VTP_{(z_0,\dots,z_n)} = \langle i\,(z_0, \dots, z_n)\rangle_{\mathbb{R}}$$

and

$$\omega_A = \sum_{i=0}^{n} \overline{z_i}\, dz_i$$

$$= i\, \mathrm{Im}\left(\sum_{i=0}^{n} \overline{z_i}\, dz_i\right)$$

and

$$H_A = \langle (z_0, \dots, z_n)\rangle_{\mathbb{C}}^{\perp_{\mathbb{C}}}$$