Introduction to Floer homotopy theory

Markus Upmeier

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Abstract

1 Morse–Bott functions

Definition 1.1. A smooth function $f\colon X \to \mathbb{R}$ is *Morse–Bott* if

$$C(f) = \{x \in X \mid df(x) = 0\}$$

is a submanifold of X, possibly disconnected and with connected components $C_k(f)$ of different dimensions. For each $x \in C(f)$ the *Hessian*

$$\mathrm{Hess}_x(v, w) = v_x(w(f)) = w_x(v(f))$$

is required to be non-degenerate on the normal space $N_x = T_xX/T_xC(f)$.

As the Hessian is a non-degenerate symmetric bilinear form, it splits the normal bundle over $C_k(f)$ as

$$N_k = N_k^- \oplus N_k^+$$

into positive and negative eigenspaces. The rank of the vector subbundle N_k^- is called the *index* of the *critical submanifold* $C_k(f)$.

We have

$$\mathrm{rk}\, N_k^- + \mathrm{rk}\, N_k^+ + \dim C_k(f) = \dim X. \tag{1.1}$$

Example 1.2. $f(x, y, z) = x^2 - z^2$ on $X = \mathbb{R}^3$.

Example 1.3. The height function of a torus lying on the side.

We will suppose that X is a compact finite-dimensional Riemannian manifold. The negative gradient vector field may then be integrated to a flow

$$\Psi\colon \mathbb{R} \longrightarrow \mathrm{Diff}(X), \quad t \longmapsto \Psi_t.$$

Hence the (downward) gradient flow line through $x \in X \setminus C(f)$ is

$$\gamma(t) = \Psi_t(x).$$

Definition 1.4. The *stable* (+) and *unstable* (−) sets of $C_k(f)$ are

$$W_k^{\pm} = \Big\{ x \in X \;\Big|\; \lim_{t\to\pm\infty} \Psi_t(x) \in C_k(f) \Big\}. \tag{1.2}$$

Theorem 1.5 (Morse–Bott Lemma)**.** *There exists a tubular neighborhood $N_k \cong U_k \subset X$ of $C_k(f)$ on which f can be identified with the quadratic form $v \mapsto \mathrm{Hess}(f)(v,v)$ on N_k.*

This implies that $W_k^{\pm} \cong N_k^{\pm}$ is a fiber bundle over $C_k(f)$ [picture] and that they are submanifolds of X of dimensions

$$\dim W_k^{\pm} = \dim C_k(f) + \mathrm{rk}\, N_k^{\pm}.$$

Definition 1.6. A Morse–Bott function satisfies *Smale transversality* if all stable and unstable manifolds intersect transversally $W_k^+ \pitchfork W_\ell^-$.

2 Flow category

For simplicity, we now assume that our indexing has been chosen so that

$$\mathrm{rk}\, N_k^- = k.$$

Definition 2.1. Let $C_{j,i}$ be the *space of broken trajectories* $(\gamma_0, \dots, \gamma_p)$ where $p \geqslant 1$ and $\gamma_i \colon \mathbb{R} \to X$ is a downward gradient flow line of f with

$$\lim_{t\to\pm\infty} \gamma_i(t) = x_i^{\pm}, \qquad x_i^+ = x_{i+1}^-, \qquad x_0^- \in C_j, x_p^+ \in C_i.$$

The *unbroken* trajectories can be identified with points in $W_j^- \cap W_i^+$ modulo translation, so by transversality their dimension is

$$\begin{aligned} \dim W_j^- \cap W_i^+ - 1 &= \dim W_j^- + \dim W_i^+ - \dim X - 1 \\ &= j + \dim C_j + (\dim X - i) - \dim X - 1 \\ &= j - i - 1 + \dim C_j. \end{aligned}$$

Theorem 2.2. *Assuming Smale transversality, the space $C_{j,i}$ of broken trajectories is still a manifold with corners. Above, we have described the open stratum.*

Definition 2.3. The *flow category* of $f \colon X \to \mathbb{R}$ has object space

$$\bigsqcup\nolimits_k C_k(f).$$

The morphism space between $x \in C_j(f)$ and $y \in C_i(f)$ is $C_{j,i}$.

The idea of Floer homotopy theory is that the (stable) homotopy type of X can be reconstructed from the flow category.

Before we explain this, we study the geometry of the space $C_{j,i}$ in more detail. Let $\pi_i\colon C_{j,i} \to C_i$ and $\pi_j\colon C_{j,i} \to C_j$ be the projections onto the endpoints. Elements $\gamma \in \pi_j^{-1}(x)$ are completely determined by a point $v \in S(N_j^-|_x)$. Given γ with $\pi_j(\gamma) = x$ and $\pi_i(\gamma) = y$, the normal bundle of $\pi_j^{-1}(x)$ in $S(N_j^-|_x)$ is $N_i^-|_y$. Hence

$$TS(\pi_j^* N_j^-) = \mathrm{Ker}(d\pi_j) \oplus \pi_i^*(N_i^-).$$

Adding a trivial bundle gives a *framing*

$$\pi_j^* N_j^- = \mathrm{Ker}(d\pi_j) \oplus \underline{\mathbb{R}} \oplus \pi_i^*(N_i^-). \tag{2.1}$$

3 Building a homotopy type

Let X_k be the union of all broken downward gradient trajectories emanating on $C_k(f)$. This defines a filtration

$$X_0 \subset X_1 \subset \cdots \subset X_n = X$$

whose successive quotients $Y_k = X_k/X_{k-1}$ are the Thom spaces $\mathrm{M}(N_k^-)$. Whenever we have a filtration, it follows from the Puppe sequence that we can recover the *stable* homotopy type from these quotients Y_k together with certain maps that specify how to glue the union as

$$\Sigma^n X_n = \Sigma^n X_0 \cup C(\Sigma^{n-1} Y_1) \cup \cdots \cup C^n(Y_n).$$

The authors encode this information as a topological functor

$$Z\colon \mathcal{J}_0^n \longrightarrow \{\text{compact pointed spaces}\}$$

on a category with $\mathrm{Ob}(\mathcal{J}_0^n) = \{0, \ldots, n\}$ and $\mathrm{Hom}_{\mathcal{J}_0^n}(j,i) = \mathcal{S}^{j-i-1}$. Thus Z is enough to reconstruct the stable homotopy type. They define $Z(k) = \Sigma^{n-k}\mathrm{M}(N_k^-)$ to be the Thom spaces and need to construct

$$S^{j-i-1} \wedge Z(j) \longrightarrow Z(i). \tag{*}$$

When f satisfies Smale transversality, $F(j,i)$ is a compact manifold with corners and we have projections

$$F_j \xleftarrow{\pi_j} F(j,i) \xrightarrow{\pi_i} F_i.$$

Combining the embedding $F(j,i) \hookrightarrow N_j^-$ with a boundary defining function $F(j,i) \to \mathbb{R}_+^k$, gives a 'neat' embedding $F(j,i) \hookrightarrow N_j^- \times \mathbb{R}_+^k$ with normal bundle $\nu_{j,i} \cong \pi_j^*(N_i^-) \times \mathbb{R}^{j-i}$ (this follows from (**??**)) and hence a diagram

$$N_j^- \times \mathbb{R}_+^k \longleftarrow \nu_{j,i} \longrightarrow N_i^- \times \mathbb{R}^{j-i} \tag{**}$$

of an open embedding (neat submanifolds have tubular neighborhoods) and a proper map. Passing to one-point compactifications gives the required

$$\Sigma^k \mathrm{M}(N_j^-) \longrightarrow \mathrm{M}(\nu_{j,i}) \longrightarrow \Sigma^{j-i}\mathrm{M}(N_i^-). \tag{***}$$

When we only have *virtual* bundles N_i^-, N_j^- but still a framing, the same methods works if we replace the codomain of Z by the stable homotopy category. Without a framing, we only retain the first map in (**) and (***).

Assuming all vector bundles have complex structures, for every complex-oriented homology theory we can define

$$E_{*-k-j}(F_j) \cong E_*(\Sigma^k \mathrm{M}(N_j^-)) \longrightarrow E_*(\mathrm{M}(\nu_{j,i})) \cong E_{*-j}(F_{j,i}) \xrightarrow{(\pi_i)_*} E_{*-j}(F_i)$$

using the Thom isomorphism. The Thom class $\Sigma^{\infty-j}\mathrm{M}(\nu_{j,i}) \to \mathrm{MU}$ combined with the projection $\nu_{j,i} \to F_{j,i} \to F_i$ yields the second map in

$$\Sigma^{\infty+k-j}\mathrm{M}(N_j^-) \longrightarrow \Sigma^{\infty-j}\mathrm{M}(\nu_{j,i}) \longrightarrow \mathrm{MU} \wedge \Sigma^\infty_+ F_i \xleftarrow{\sim} \mathrm{MU} \wedge \Sigma^{\infty-i}\mathrm{M}(N_i^-).$$

The last arrow is induced in the same way by the Thom class, and is a π_*-isomorphism by the Thom isomorphism theorem. Hence we get

$$\mathcal{S}^{j-i-1} \wedge \Sigma^{\infty+n-j}\mathrm{M}(N_j^-) \longrightarrow \mathrm{MU} \wedge \Sigma^{\infty+n-i}\mathrm{M}(N_i^-)$$

in the stable homotopy category. From such a functor Z one may still extract a ‘pro-spectrum’.

References

[1] R.L. Cohen, J.D.S. Jones and G.B. Segal, *Floer’s infinite-dimensional Morse theory and homotopy theory*, pp.297–325 in *The Floer memorial volume*, Progr. Math. **133**, Birkhäuser, Basel, 1995.