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Monopole Homology

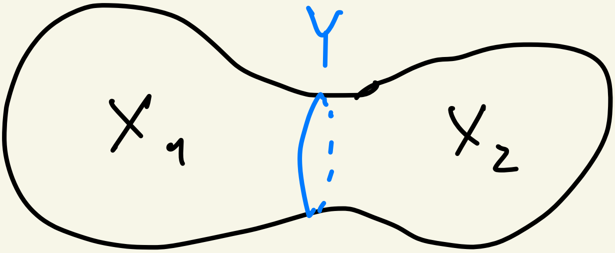
I. Gluing SW $\rightsquigarrow$ Floer theory.

II. Reducibles & blow-up.

III. Cobordism maps
& relative invariants.

I. From SW gluing to Floer theory

X



$$SW(X) = ?$$

or: $SW(\tilde{X} = X_1 \cup_{f: Y \to Y} X_2) = ?$

$$\mathcal{M}(X) \simeq \underbrace{\mathcal{M}(X_1)}_{\infty\text{-dim}} \times_{\underbrace{\mathcal{B}(Y)}_{\infty\text{-dim}}} \underbrace{\mathcal{M}(X_2)}_{\infty\text{-dim}}$$

soln's to SW

Hope : $\# \mathcal{M}(X) =$

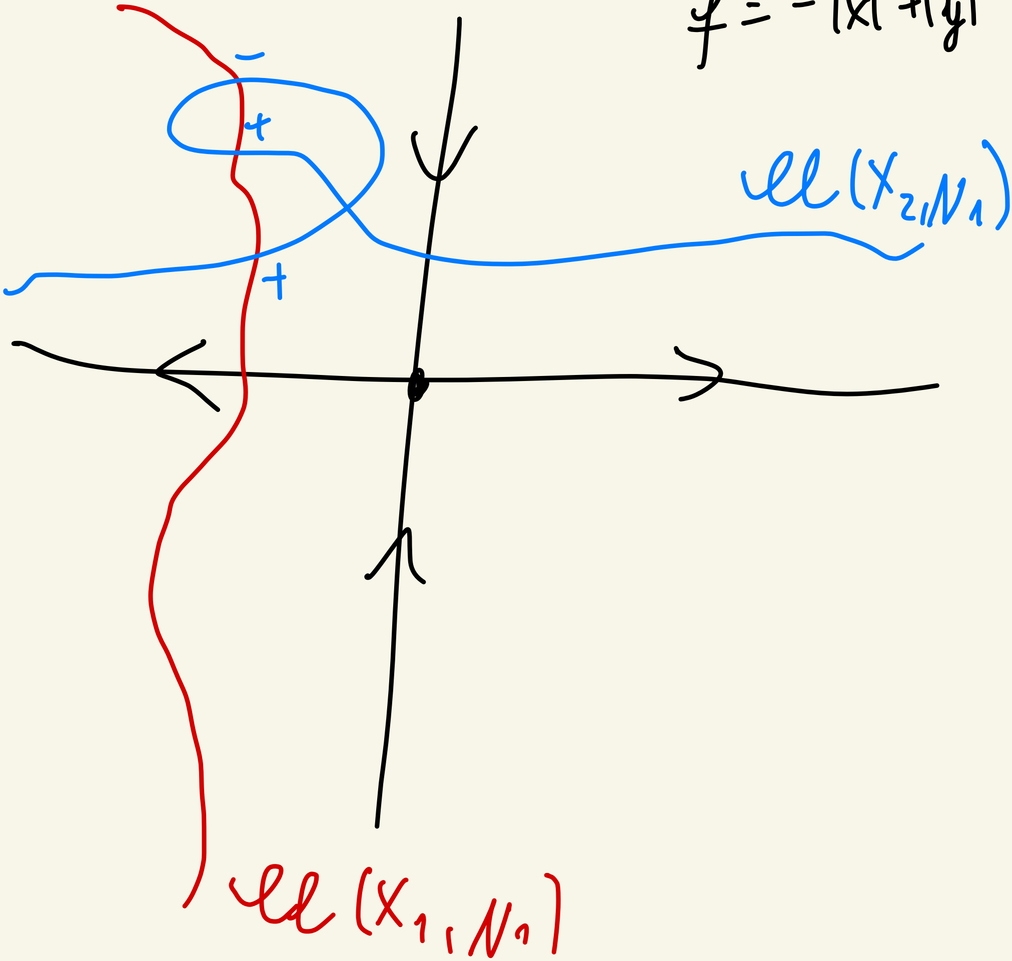
" = " $\langle [\mathcal{M}(X_1)], [\mathcal{M}(X_2)] \rangle$

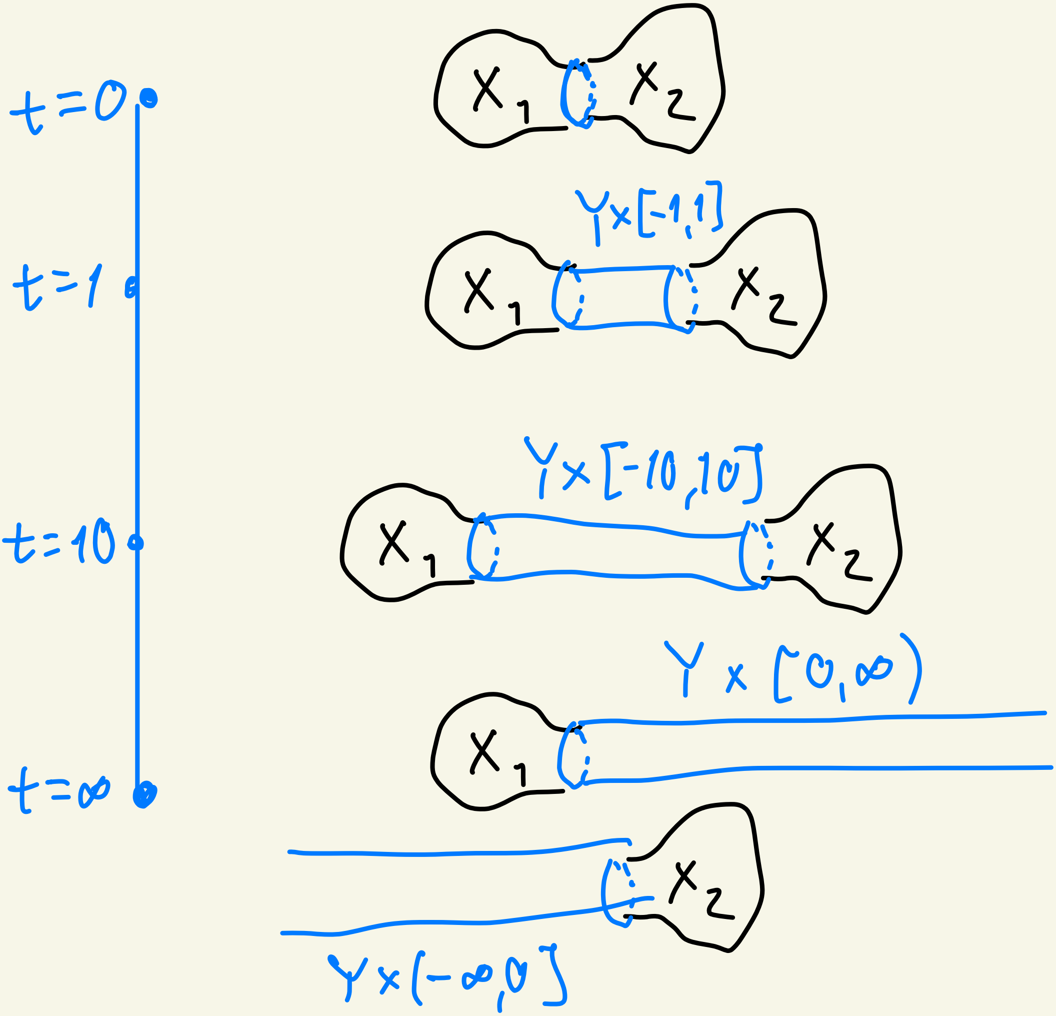
in "$HF^*(\mathcal{C}(Y))$"

$(A = A_0 + \alpha, \ \Phi)$

$\Gamma(\Omega^1(X; i\mathbb{R}))$ $\quad$ $\Gamma(S_X)$

$f = -|x|^2 + |y|^2$





$$SW_4: \begin{cases} \frac{1}{2}F_A^+ + \rho^{-1}(\Phi\Phi^*) = 0 \\ \not{D}_A \Phi = 0 \end{cases} \quad \text{on } X^4$$

on $Y\times\mathbb{R}_t$, assume t-indep

$$SW_3: \begin{cases} \frac{1}{2}F_B + \rho^{-1}(\Phi\Phi^*)_0 = 0 \\ \not{D}_B \Phi = 0 \end{cases}$$

NOT assuming t-indep ?

$$\left\{ SW_4(A, \Phi) = 0 \text{ on } Y \times \mathbb{R} \right\} \Big/ \text{gauge}$$

$/t$-transl.

$$\longleftrightarrow \left\{ \partial_t (B_t, \Phi_t) = -\operatorname{grad}_{L^2} \mathcal{L}(B_t, \Phi_t) \right\} \Big/ \text{gauge}$$

$$\begin{cases} \bullet\ A = dt + \pi^* B_t \\ \bullet\ \Phi|_{Y \times \{t\}} = \Phi_t \end{cases}$$

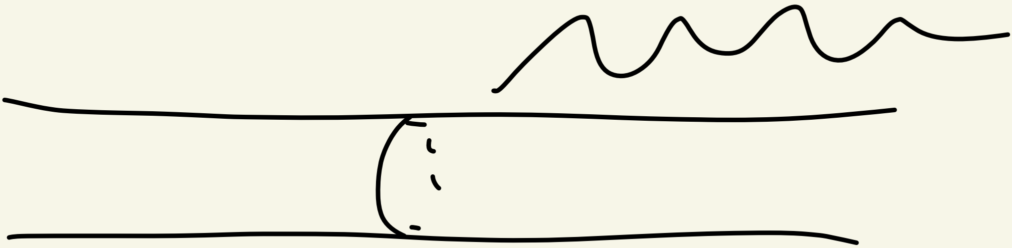
$$\mathcal{L}(B, \Phi) = -\frac{1}{2} \int b \wedge (F_{B_0^t} + db) + \frac{1}{2} \int_Y \langle \not{D}_B \Phi, \Phi \rangle$$

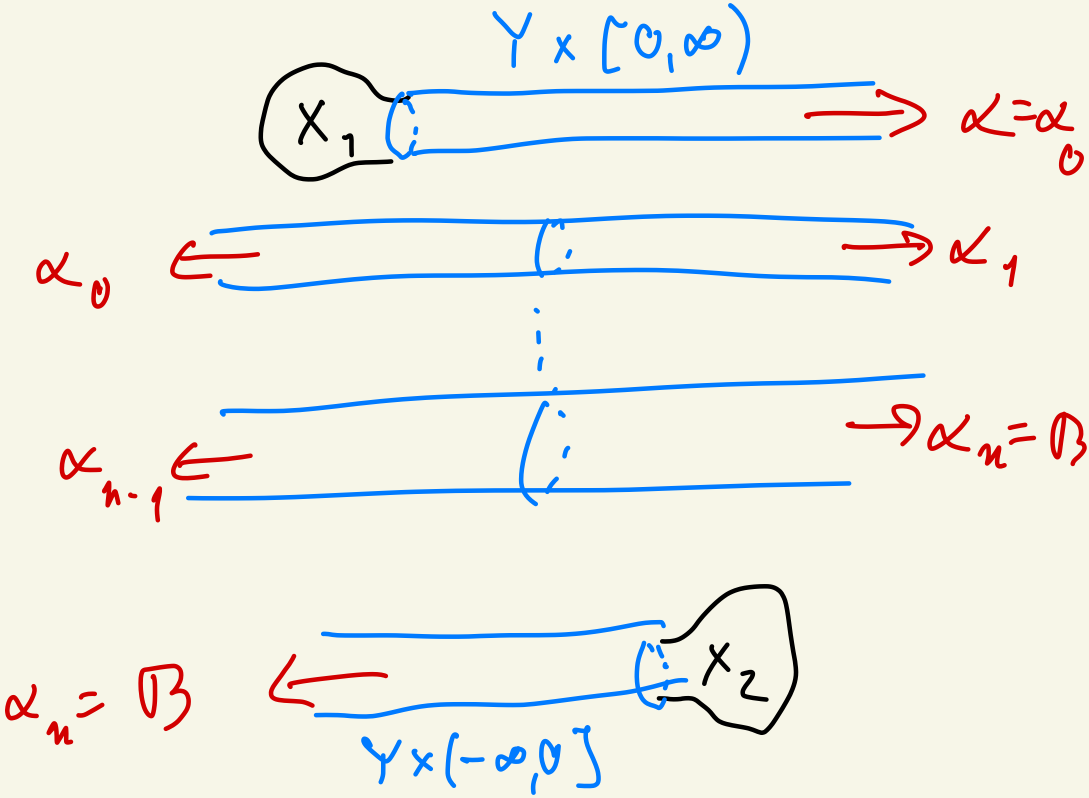
$B = B_0 + b$

$$\nabla_{L^2} \mathcal{L} = \begin{pmatrix} \frac{1}{2} \star F_B + \rho^{-1}(\Phi \Phi^*)_0 \\ \not{D}_B \Phi \end{pmatrix}$$

$SW_4 = 0$ on $Y \times \mathbb{R}$

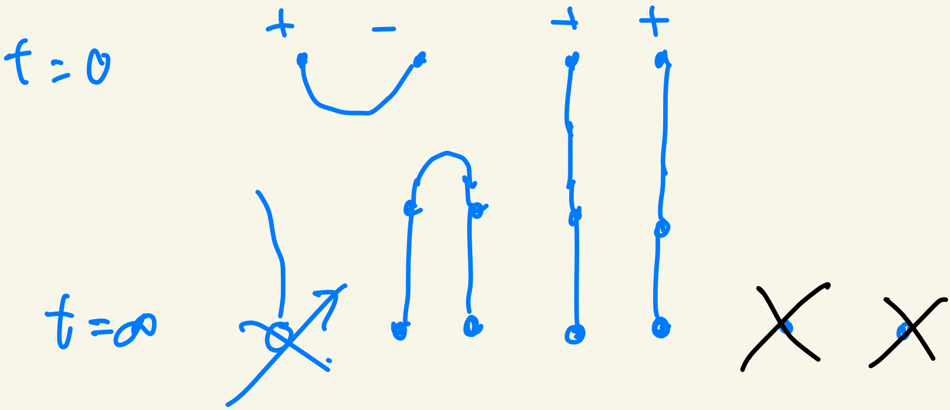
- crit pts of $\mathcal{L}$ $\quad$ $\mathrm{Crit}\,\mathcal{L}$
  (t-indep. solns)
- (assuming fin. en.)
  trajectories of $-\nabla\mathcal{L}$
  between $\mathrm{Crit}\,\mathcal{L}$



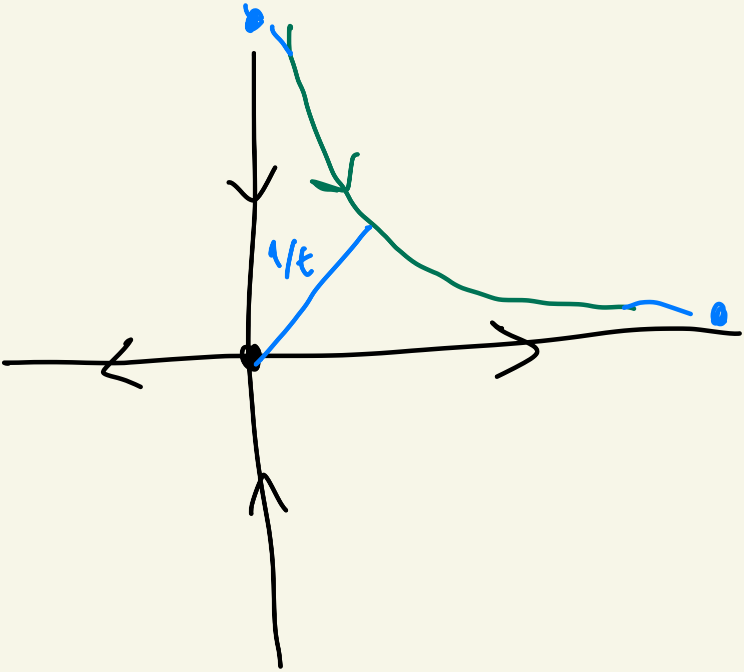


$Y \times [0,\infty)$
X_1
$\alpha = \alpha_0$
α_0
α_1
α_{n-1}
$\alpha_n = \beta$
$\alpha_n = \beta$
X_2
$Y \times (-\infty, 0]$

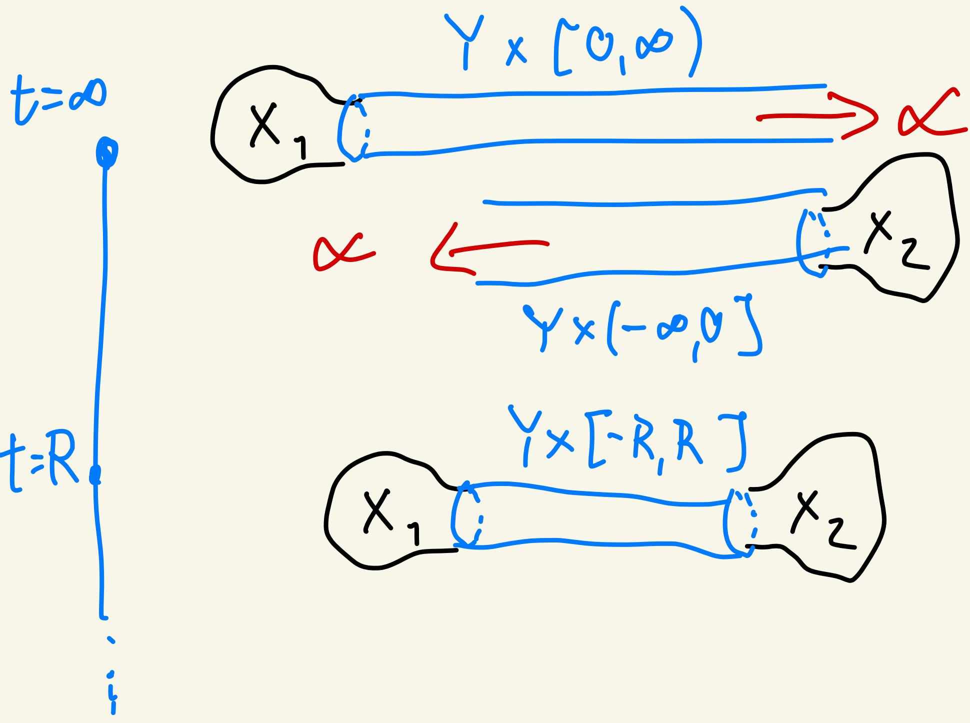
$\# \mathcal{M}(X, N)$

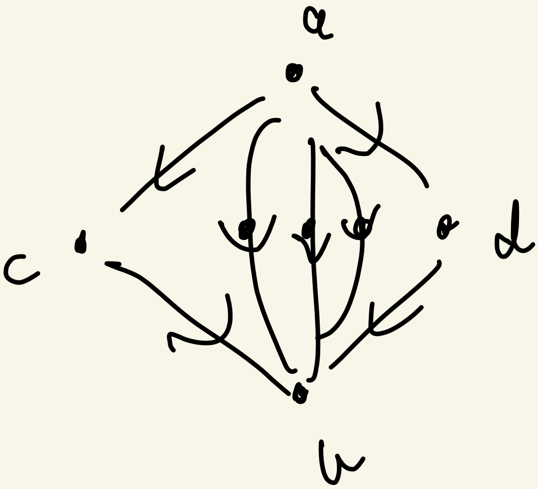


$t = 0$
$+$
$-$
$+$
$+$
$t = \infty$

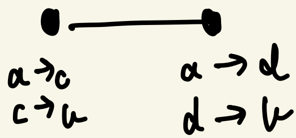


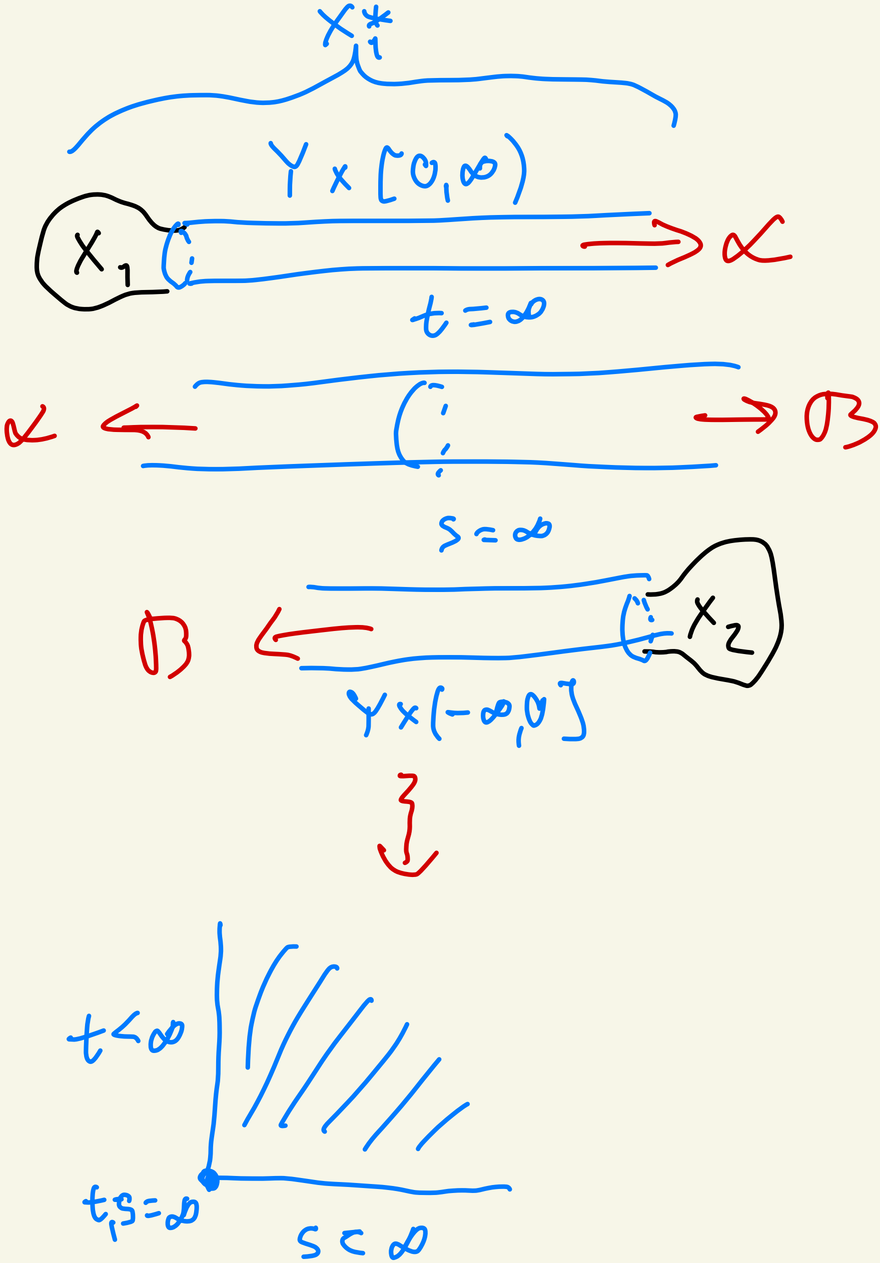
Assume α_i's are irreducible.





$\mathrm{ul}(a,b) =$ 

$\widetilde{\mathrm{ul}}(a,b) =$ 



X_1^*
$Y \times [0,\infty)$
X_1
α
$t=\infty$
α
β
$s=\infty$
β
X_2
$Y \times (-\infty,0]$
$t<\infty$
$t,s=\infty$
$s<\infty$

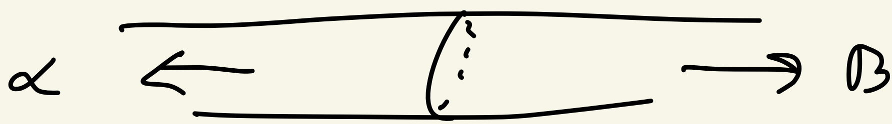
$$[\mathcal{M}(X_1)] = \sum \# \mathcal{M}(X_1^*, \alpha) \cdot \langle \alpha \rangle$$

$$\in \bigoplus_{\alpha \in \mathrm{Crit}\, \mathcal{L} / \mathcal{G}} \mathbb{Z} \langle \alpha \rangle$$

$$\parallel$$

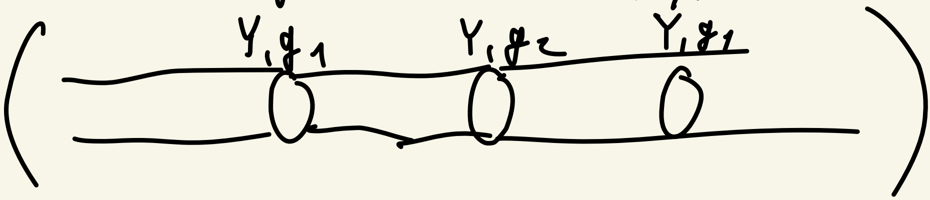
$$C_*(Y, N)$$

$$d\langle \alpha \rangle = \sum \# \mathcal{M}(\alpha, \beta) \langle \beta \rangle$$



if NO REDS

$\Rightarrow$ get $HM_*(Y, N)$



II. Reducibles

$$\mathcal{C}(Y,\mathcal{N}) = \mathcal{B}(Y,\mathcal{N}) \times \Gamma(S_Y)$$

$\circlearrowleft S^1_{\text{trivial}}$ $\circlearrowleft S^1_{\text{cx mult}}$

$$\mathcal{G}^0(Y) = \{u \in L^2_k(Y, S^1) \mid u(y_0) = 1\}$$

$$0 \to \underbrace{\mathcal{G}^0(Y)}_{\text{acts freely on } \mathcal{C}(Y,\mathcal{N})} \to \mathcal{G}(Y) \to S^1 \to 0$$

$$B^0(Y,\mathcal{N}) = \mathcal{C}(Y,\mathcal{N}) / \mathcal{G}^0(Y)$$

$\circlearrowleft S^1$

$$\simeq \tilde{\mathcal{B}}(Y,\mathcal{N}) \times \Gamma(S_Y)$$

comes in "Coul gauge"

$\tilde{\mathcal{B}}(Y,\mathcal{N}) \times \{0\}$ fixed by S^1

$N \circlearrowleft S^1$ semi-free

$M = N^\sigma / S^1$, w/ ∂

$\begin{array}{c} \uparrow \\ N^\sigma \end{array}$ blow-up of N along N^{S^1}

$S^1 \circlearrowright N^\sigma \simeq N \times ES^1 \circlearrowleft S^1$

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then

$$\begin{array}{ccc} H_*(M) & \sim & H_*^{S^1}(N) \\ \downarrow & & \downarrow \\ H_*(M, \partial M) & \sim & cH_*^{S^1}(N) \\ \downarrow & & \downarrow \\ H_*(\partial M) & \sim & tH_{*-1}^{S^1}(N) \end{array}$$

Borel

coBorel

Tate

$\mathbb{R}_{\geq 0} \times H$

$\mathbb{C}^n \times \mathbb{C}^m$

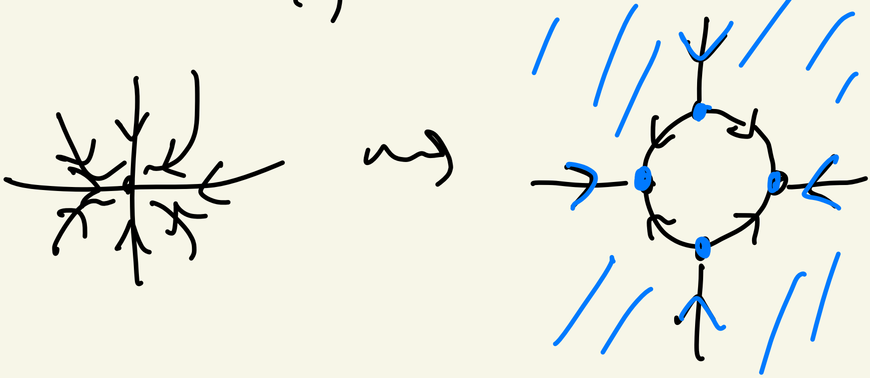
$$\mathcal{L} = -|x_1|^2 - 2|x_2|^2 - \dots$$
$$+ |y_1|^2 + 2|y_2|^2 + \dots$$

Blow-up:

$$\{(v \in S^{2n+2m-1}, r \in [0,\infty))\} / S^1$$

$$\overset{htp}{\sim} \mathbb{CP}^{n+m-1}$$

$\nabla \mathcal{L}|_{\mathbb{C}^n \times \mathbb{C}^m \setminus \{0\}}$ extends! to



$$\mathcal{B}^\sigma(Y, N) =$$

$$= \left\{ \left(B_0 + b, \ \Phi, \ s \right) \right\}_{\substack{s \geq 0 \\ \|\Phi\|_{L^2} = 1}} \Big/ \mathcal{G}(Y)$$

$$\partial \mathcal{B}^\sigma(Y, N) =$$

$$= \left\{ \left(B_0 + b, \ \Phi, \ 0 \right) \right\} \Big/ \mathcal{G}(Y)$$

$$\operatorname{Crit}\mathcal{L} = C^{o} \cup C^{u} \cup C^{s}$$

$\cap$

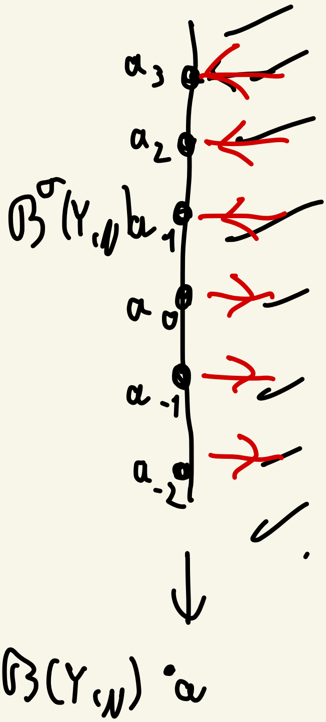
$\mathcal{B}^{\sigma}(Y,N)$

C^{o}: irred; C^{u}: red unst.; C^{s}: red st.

$\mathcal{B}(Y,N) \ni a$ irreducible

$\rightsquigarrow$ a irr $\in \mathcal{B}^{\sigma}(Y,N)$

a reducible



$a_i \overset{\mathcal{U}}{\rightsquigarrow} a_{i-1}$

is $\mathbb{CP}^1$

$\left(\mathcal{U}_+ a_i = a_{i-1} \right)$

"from" $\hat{C}_* = C^0_* \oplus C^u_{*+1}$

"to" $\check{C}_* = C^0_* \oplus C^s_*$

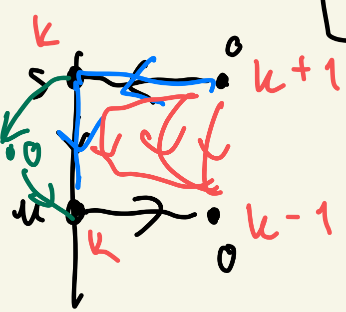
"bar" $\bar{C}_* = C^u_* \oplus C^s_*$

$$\bar{d} = \begin{bmatrix} \bar{d}^u_u & \bar{d}^s_u \\ \bar{d}^u_s & \bar{d}^s_s \end{bmatrix}$$

$$\# \overline{\mathcal{M}}(a_u, a_s)$$

ONLY REDUCIBLE

$$\hat{d} = \begin{bmatrix} \pm d^0_0 & \pm d^u_0 \\ \cancel{d^0_u} \pm \bar{d}^s_u d^0_s & \pm \bar{d}^u_u \pm \bar{d}^s_u d^u_s \end{bmatrix}$$



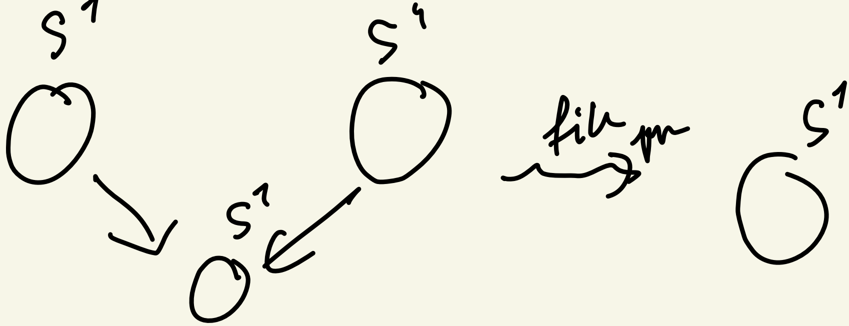
$\bullet \xrightarrow{o} u$

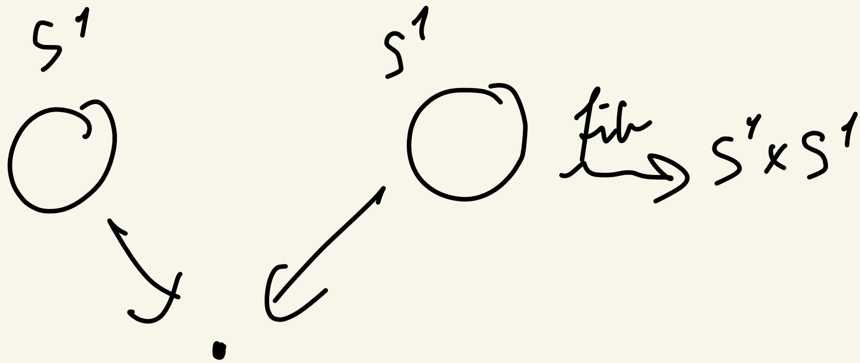
$s \xrightarrow{o} \bullet$

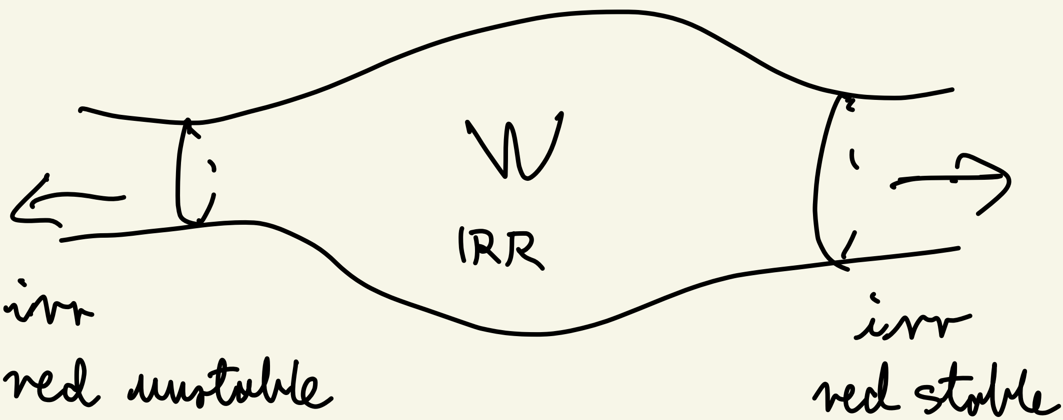
$\bullet \xrightarrow{\text{red}} o \qquad o \xrightarrow{\text{red}} \bullet$

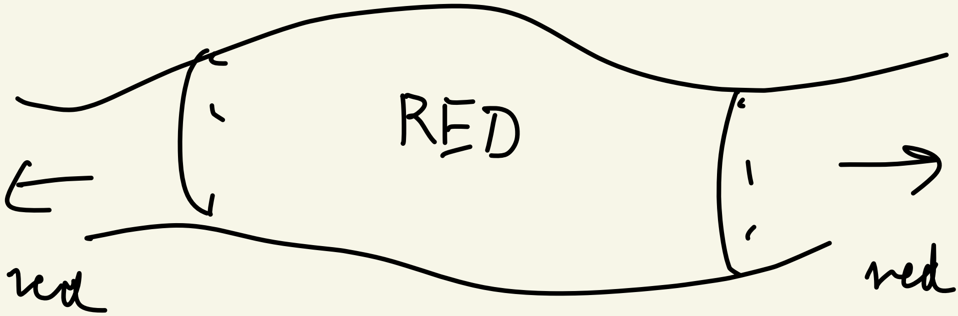
$$\to \widehat{HM}_{*-1}(Y) \to \overline{HM}_*(Y) \to \check{HM}_*(Y) \to$$

MONOPOLE HOMOLOGY









reds $\longleftrightarrow$ flat conns