

RATIONAL HOMOLOGY RIBBON COBORDISM IS A PARTIAL ORDER

STEFAN FRIEDL, FILIP MISEV, AND RAPHAEL ZENTNER

ABSTRACT. We show that ribbon rational homology cobordism is a partial order within the class of irreducible 3-manifolds. This makes essential use of the methods recently employed by Ian Agol to show that ribbon knot concordance is a partial order.

1. INTRODUCTION

It has recently been proved by Agol that ribbon concordance of knots is a partial order, [1]. In this paper we prove an analogous statement for the preorder on irreducible, closed, connected, oriented 3-manifolds that is given by rational homology cobordisms.

Let Y_0 and Y_1 be closed, connected, oriented 3-manifolds. We say that a compact, connected, oriented, smooth 4-manifold W is a *cobordism from Y_0 to Y_1* if $\partial W = Y_1 \sqcup (-Y_0)$. A cobordism W is called a *\mathbb{Q} -homology cobordism from Y_0 to Y_1* if the inclusions of Y_0 and Y_1 into W both induce an isomorphism in homology with \mathbb{Q} -coefficients. Finally we say that W is a *ribbon cobordism from Y_0 to Y_1* if W is built from $Y_0 \times [0, 1]$ by attaching only 1-handles and 2-handles.

We write $Y_1 \geq Y_0$ if there exists a ribbon \mathbb{Q} -homology cobordism from Y_0 to Y_1 . Clearly this defines a preorder, i.e. the relation \geq is reflexive and transitive. It is less clear whether this is anti-symmetric and so defines a partial order, i.e. if $Y_0 \geq Y_1$ and $Y_1 \geq Y_0$, are then Y_0 and Y_1 homeomorphic?

The following conjecture has been formulated by Daemi, Lidman, Vela-Vick, and Wong [3, Conjecture 1.1]:

Conjecture 1.1. *The preorder on the set of homeomorphism classes of closed, connected, oriented 3-manifolds given by ribbon \mathbb{Q} -homology cobordism is a partial order, i.e. if one has $Y_0 \geq Y_1$ and $Y_1 \geq Y_0$ then Y_0 and Y_1 are homeomorphic.*

We prove this conjecture for irreducible 3-manifolds.

Theorem 1.2. *The preorder \geq is a partial order on the set of homeomorphism classes of irreducible, closed, connected, oriented 3-manifolds. In particular, if $Y_0 \geq Y_1$ and $Y_1 \geq Y_0$ then Y_0 and Y_1 are homeomorphic.*

In fact for aspherical 3-manifolds we can prove a refinement.

Theorem 1.3. *The preorder \geq is a partial order on the set of orientation-preserving homeomorphism classes of aspherical, closed, connected, oriented 3-manifolds. In particular, if $Y_0 \geq Y_1$ and $Y_1 \geq Y_0$ then there exists an orientation-preserving homeomorphism from Y_0 to Y_1 .*

It is not clear to us whether the conclusion of Theorem 1.3 also holds for irreducible 3-manifolds that are not aspherical. This leads us to the following question.

Question 1.4. *Does there exist a spherical oriented 3-manifold Y with $Y \leq -Y$?*

In the proofs of Theorems 1.2 and 1.3 we make essential use of the methods employed by Agol to prove that ribbon concordance is a partial order. More precisely, Agol's methods apply to the situation we are considering and provide us with the following theorem.

Theorem 1.5. *Suppose Y is a closed, connected, oriented 3-manifold. Suppose W is a ribbon \mathbb{Q} -homology cobordism from $Y_- \cong Y$ to $Y_+ \cong Y$, i.e. $Y_+ \geq Y_-$ (and so from Y to itself). Then the inclusion $\iota_+ : Y_+ \rightarrow W$ induces an isomorphism on fundamental groups.*

Using Theorem 1.5 one can easily prove the following corollary which is the key ingredient in the proofs of Theorems 1.2 and 1.3.

Corollary 1.6. *Suppose W_0 is a ribbon \mathbb{Q} -homology cobordism from Y_0 to Y_1 , so that $Y_1 \geq Y_0$. Suppose that W_1 is a ribbon \mathbb{Q} -homology cobordism from Y_1 to Y_0 , so that $Y_0 \geq Y_1$. Then the injection $\iota_0 : Y_1 \rightarrow W_0$ induces an isomorphism of fundamental groups, and likewise for the injection $\iota_1 : Y_0 \rightarrow W_1$.*

Remark. We use the convention of [3], in which Conjecture 1.1 has been formulated, where $Y_1 \geq Y_0$ means that there is a ribbon cobordism from Y_0 to Y_1 , as defined above. We would like to point out that this may result in some confusion when comparing with Gordon's initial convention in [5] and the one employed by Agol [1], where a ribbon concordance goes from K_1 to K_0 if the exterior $E(C)$ of the concordance C is obtained by adding only 1-handles and 2-handles to $E(K_0) \times [0, 1]$. In other words, in their convention, a ribbon cobordism goes from the more complex object to the simpler one.

Outline. In the second section we will state and prove some auxiliary results that we will need for the proof of our main results, Theorems 1.2 and 1.3. In

the third section we provide the proof of Theorem 1.3. Afterwards in the fourth section we will use Theorem 1.3 as an ingredient in our proof of Theorem 1.2. Finally the fifth section contains a sketch of proof of Theorem 1.5 which is almost verbatim identical to Agol’s proof in the context of knot complements, together with the proof of Corollary 1.6.

Acknowledgments. SF and FM were supported by the SFB 1085 “higher invariants” at the University of Regensburg, funded by the DFG. RZ is thanking the DFG for support through the Heisenberg program. We also wish to thank Ederson Dutra and Lukas Lewark for helpful conversations, and Lukas Lewark for comments on a preliminary version of this article.

2. PREPARATIONS

In this short section we collect a few basic facts that we will use in the proof of Theorem 1.2. The following appears as [3, Proposition 2.1].

Proposition 2.1 (Daemi, Lidman, Vela-Vick, Wong). *Let Y_0 and Y_1 be closed, connected, oriented 3-manifolds and let W be a cobordism from Y_0 to Y_1 . We denote by $\iota_0: Y_0 \rightarrow W$ and $\iota_1: Y_1 \rightarrow W$ the obvious inclusion maps.*

- (1) *If W is a ribbon cobordism, then the map $(\iota_1)_*: \pi_1(Y_1) \rightarrow \pi_1(W)$ is an epimorphism.*
- (2) *If W is a ribbon \mathbb{Q} -homology cobordism, then the map $(\iota_0)_*: \pi_1(Y_0) \rightarrow \pi_1(W)$ is a monomorphism.*

By definition, a ribbon cobordism W from Y_0 to Y_1 is obtained from $Y_0 \times [0, 1]$ by attaching 1-handles and 2-handles. Flipping W upside down, we see that W can equivalently be viewed as $Y_1 \times [0, 1]$ with some 2-handles and 3-handles attached. This immediately implies the first statement. For the second statement, Daemi, Lidman, Vela-Vick, and Wong use Theorem 2 in Gerstenhaber-Rothaus’ work [4], much in the same way as initially done by Gordon in [5, Proof of Lemma 3.1], using the residual finiteness of 3-manifold groups.

The next proposition gives us some homological information about \mathbb{Q} -homology cobordisms.

Proposition 2.2. *Let Y_0 and Y_1 be closed, connected, oriented 3-manifolds. We consider their fundamental classes $[Y_0] \in H_3(Y_0; \mathbb{Z})$ and $[Y_1] \in H_3(Y_1; \mathbb{Z})$. Let W be a cobordism from Y_0 to Y_1 . If W is a \mathbb{Q} -homology cobordism, then the maps $(\iota_0)_*: H_3(Y_0; \mathbb{Z}) \rightarrow H_3(W; \mathbb{Z})$ and $(\iota_1)_*: H_3(Y_1; \mathbb{Z}) \rightarrow H_3(W; \mathbb{Z})$ are both isomorphisms. Furthermore $(\iota_0)_*([Y_0]) = (\iota_1)_*([Y_1]) \in H_3(W; \mathbb{Z})$.*

Proof. Let $i \in \{0, 1\}$. We have the following commutative diagram:

$$\begin{array}{ccccccc} \dots & \rightarrow & H_4(W, Y_i; \mathbb{Z}) & \rightarrow & H_3(Y_i; \mathbb{Z}) & \xrightarrow{(\iota_i)_*} & H_3(W; \mathbb{Z}) & \rightarrow & H_3(W, Y_i; \mathbb{Z}) & \rightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & H_3(Y_i; \mathbb{Q}) & \xrightarrow[\cong]{(\iota_i)_*} & H_3(W; \mathbb{Q}) & & & & & & \end{array}$$

Since W is a \mathbb{Q} -homology cobordism we know that the bottom horizontal map is an isomorphism. It follows from this observation and the universal coefficient theorem that the kernel and the cokernel of the map $(\iota_i)_*: H_3(Y_i; \mathbb{Z}) \rightarrow H_3(W; \mathbb{Z})$ are finite. But by Poincaré-Lefschetz duality and the universal coefficient theorem we see that

$$\begin{aligned} H_3(W, Y_i; \mathbb{Z}) &\cong H^1(W, Y_{1-i}; \mathbb{Z}) \cong \text{Hom}(H_1(W, Y_{1-i}; \mathbb{Z}), \mathbb{Z}) \\ &\quad \oplus \text{Ext}(H_0(W, Y_{1-i}; \mathbb{Z}), \mathbb{Z}) \end{aligned}$$

is torsion-free. Similarly, we see that $H_4(W, Y_i; \mathbb{Z})$ is torsion free, since

$$H_4(W, Y_i; \mathbb{Z}) \cong H^0(W, Y_{1-i}; \mathbb{Z}) \cong \text{Hom}(H_0(W, Y_{1-i}; \mathbb{Z}), \mathbb{Z}).$$

In summary, we see that the two maps $(\iota_0)_*: H_3(Y_0; \mathbb{Z}) \rightarrow H_3(W; \mathbb{Z})$ and $(\iota_1)_*: H_3(Y_1; \mathbb{Z}) \rightarrow H_3(W; \mathbb{Z})$ are isomorphisms.

It remains to show that $(\iota_0)_*([Y_0]) = (\iota_1)_*([Y_1]) \in H_3(W; \mathbb{Z})$. We consider the long exact sequence of the pair $(W, Y_0 \cup Y_1)$.

$$\dots \rightarrow H_4(W, Y_0 \cup Y_1; \mathbb{Z}) \xrightarrow{\partial} H_3(Y_0 \cup Y_1; \mathbb{Z}) \rightarrow H_3(W; \mathbb{Z}) \rightarrow \dots$$

It is well-known that $\partial([W]) = [\partial W]$. Since W is a cobordism we have $\partial W = Y_1 \sqcup (-Y_0)$, in particular $[\partial W] = [Y_1] - [Y_0]$. Since the sequence is exact we see that the image of $[Y_1] - [Y_0]$ is zero in $H_3(W; \mathbb{Z})$, i.e. we have $(\iota_0)_*([Y_0]) - (\iota_1)_*([Y_1]) = 0 \in H_3(W; \mathbb{Z})$. \square

Lemma 2.3. *Suppose that we have a continuous map $g: X \rightarrow Z$, between path-connected topological spaces X and Z which admit the structure of a CW-complex. Then for any k the following diagram commutes:*

$$\begin{array}{ccc} H_k(X; \mathbb{Z}) & \xrightarrow{g_*} & H_k(Z; \mathbb{Z}) \\ (j_X)_* \downarrow & & \downarrow (j_Z)_* \\ H_k(\pi_1(X); \mathbb{Z}) & \xrightarrow{g_*} & H_k(\pi_1(Z); \mathbb{Z}). \end{array}$$

Here the group homology $H_i(G; \mathbb{Z})$ of a group G is defined to be the singular homology of its associated $K(G, 1)$ -space: $H_i(G; \mathbb{Z}) := H_i(K(G, 1); \mathbb{Z})$. Furthermore $j_X: X \rightarrow K(\pi_1(X), 1)$ and $j_Z: Z \rightarrow K(\pi_1(Z), 1)$ are the natural

maps of the respective spaces to the Eilenberg-MacLane space associated to their fundamental groups.

Proof. We denote by $K(g): K(\pi_1(X), 1) \rightarrow K(\pi_1(Z), 1)$ the map induced by the group homomorphism $g_*: \pi_1(X) \rightarrow \pi_1(Z)$. By construction this map induces the map

$$g_*: \pi_1(X) \xrightarrow{\cong} \pi_1(K(\pi_1(X), 1)) \rightarrow \pi_1(K(\pi_1(Z), 1)) \xleftarrow{\cong} \pi_1(Z).$$

Therefore, the two maps $K(g) \circ j_X$ and $j_Z \circ g$ induce the same maps on fundamental group. Since their image space $K(\pi_1(Z), 1)$ is aspherical, these two maps are homotopic by Whitehead's theorem, and hence induce the same maps in homology. \square

Theorem 2.4. *Let Y_0 and Y_1 be irreducible, closed, orientable 3-manifolds and let $\alpha: \pi_1(Y_0) \rightarrow \pi_1(Y_1)$ be an isomorphism.*

- (1) *If Y_0 and Y_1 are not lens spaces, then Y_0 and Y_1 are homeomorphic.*
- (2) *If Y_0 and Y_1 are not spherical, then there exists a homeomorphism $g: Y_0 \rightarrow Y_1$ which induces α , i.e. which satisfies $g_* = \alpha: \pi_1(Y_0) \rightarrow \pi_1(Y_1)$.*

Proof. The theorem is stated as [2, Theorem 2.1.2]. It is a consequence of the Geometrization Theorem, the Mostow Rigidity Theorem, work of Waldhausen [10, Corollary 6.5], Scott [9, Theorem 3.1] and classical work on spherical 3-manifolds (see [8, p. 113]). \square

We conclude our section of preparations with the following theorem that was recently proved by Huber [7].

Theorem 2.5 (Huber). *Let $L(p_1, q_1)$ and $L(p_2, q_2)$ be lens spaces. If $L(p_1, q_1) \leq L(p_2, q_2)$, then one of the following holds:*

- (1) *the lens spaces are homeomorphic,*
- (2) *there exists an $n \geq 2$ with $L(p_1, q_1) \cong L(n, 1)$ and $p_2/q_2 \in \mathcal{F}_n$, where*

$$\mathcal{F}_n := \left\{ \frac{nm^2}{nmk+1} \mid m > k > 0, \gcd(m, k) = 1 \right\}.$$

- (3) *$L(p_1, q_1) \cong S^3$.*

3. PROOF OF THEOREM 1.3

We first provide the proof of Theorem 1.3, since the statement of Theorem 1.3 gives us also most of Theorem 1.2.

Proof of Theorem 1.3. Let Y_0 and Y_1 be aspherical, closed, connected, oriented 3-manifolds. We assume that $Y_0 \geq Y_1$ and that $Y_1 \geq Y_0$. We need to show that there exists an orientation-preserving homeomorphism from Y_0 to Y_1 .

Let W be a ribbon \mathbb{Q} -homology cobordism from Y_0 to Y_1 . We denote by $\iota_0: Y_0 \rightarrow W$ and $\iota_1: Y_1 \rightarrow W$ the obvious inclusion maps. Since we also assume that $Y_1 \geq Y_0$ we obtain from Corollary 1.6 that $(\iota_1)_*: \pi_1(Y_1) \rightarrow \pi_1(W)$ is an isomorphism. Furthermore by Proposition 2.1 (2) we know that $(\iota_0)_*: \pi_1(Y_0) \rightarrow \pi_1(W)$ is a monomorphism. We write $\alpha := ((\iota_1)_*)^{-1} \circ (\iota_0)_*: \pi_1(Y_0) \rightarrow \pi_1(Y_1)$.

Since Y_1 is aspherical there exists a map $f: Y_0 \rightarrow Y_1$ with $f_* = \alpha: \pi_1(Y_0) \rightarrow \pi_1(Y_1)$. We will show that this map f has degree equal to ± 1 . For this we consider the following diagram, which is commutative by Lemma 2.3:

$$\begin{array}{ccccc} H_3(Y_0; \mathbb{Z}) & \xrightarrow[\cong]{(\iota_0)_*} & H_3(W; \mathbb{Z}) & \xleftarrow[\cong]{(\iota_1)_*} & H_3(Y_1; \mathbb{Z}) \\ (j_0)_* \downarrow \cong & & \downarrow (j_W)_* & & \cong \downarrow (j_1)_* \\ H_3(\pi_1(Y_0); \mathbb{Z}) & \xrightarrow{(\iota_0)_*} & H_3(\pi_1(W); \mathbb{Z}) & \xleftarrow[\cong]{(\iota_1)_*} & H_3(\pi_1(Y_1); \mathbb{Z}). \end{array} \quad (1)$$

Here, $j_0: Y_0 \hookrightarrow K(\pi_1(Y_0), 1)$ is the natural map of Y_0 to the Eilenberg-MacLane space of its fundamental group, and likewise j_1 and j_W are defined.

It follows from our hypothesis that W is a \mathbb{Q} -homology cobordism and Proposition 2.2 that the horizontal maps in the first line of this diagram are isomorphisms. By hypothesis Y_0 and Y_1 are already aspherical, i.e. they are Eilenberg-MacLane spaces. Therefore, the maps $(j_0)_*$ and $(j_1)_*$ are isomorphisms.

Now by commutativity of the right square in the diagram (1), the vertical map $(j_W)_*$ induced by the inclusion map j_W must be an isomorphism. Therefore, by commutativity of the left square in this diagram, we conclude that the map $(\iota_0)_*: H_3(\pi_1(Y_0); \mathbb{Z}) \rightarrow H_3(\pi_1(W); \mathbb{Z})$ is an isomorphism.

As in the proof of Lemma 2.3 above, we denote by $K(\varphi): K(G, 1) \rightarrow K(H, 1)$ the map induced on Eilenberg-MacLane spaces by a group homomorphism $\varphi: G \rightarrow H$. In our situation, we have the two maps $f: Y_0 \rightarrow Y_1$ and $K((\iota_1)_*^{-1}) \circ K((\iota_0)_*): Y_0 \rightarrow Y_1$, where we identify Y_0 and Y_1 with Eilenberg-MacLane spaces of their respective fundamental group. Both induce the same map at the level of fundamental groups. Since Y_1 is aspherical, these maps are homotopic by Whitehead's theorem. Therefore, they induce the same map on homology. By the above observation that the bottom maps in diagram (1) are isomorphisms, we conclude that $f_*: H_3(Y_0) \rightarrow H_3(Y_1)$ is an isomorphism, and therefore f has degree ± 1 .

By a standard argument a map $f: Y_0 \rightarrow Y_1$ of degree ± 1 induces an epimorphism of fundamental groups. Since we already know that $f_* = \alpha$ is a monomorphism we see that $f_*: \pi_1(Y_0) \rightarrow \pi_1(Y_1)$ is an isomorphism. Thus it follows from Theorem 2.4 (2) that there exists a homeomorphism $g: Y_0 \rightarrow Y_1$ that induces $f_* = \alpha$. It remains to show that g is orientation-preserving. To do so we consider the commutative diagram:

$$\begin{array}{ccccc}
 H_3(Y_0; \mathbb{Z}) & \xrightarrow[\cong]{(\iota_0)_*} & H_3(W; \mathbb{Z}) & \xleftarrow[\cong]{(\iota_1)_*} & H_3(Y_1; \mathbb{Z}) \\
 (j_0)_* \downarrow \cong & & \downarrow (j_W)_* & & \cong \downarrow (j_1)_* \\
 H_3(\pi_1(Y_0); \mathbb{Z}) & \xrightarrow{(\iota_0)_*} & H_3(\pi_1(W); \mathbb{Z}) & \xleftarrow[\cong]{(\iota_1)_*} & H_3(\pi_1(Y_1); \mathbb{Z}) \\
 \text{id} \downarrow & & & & \text{id} \downarrow \\
 H_3(\pi_1(Y_0); \mathbb{Z}) & \xrightarrow{\alpha = f_* = g_*} & & & H_3(\pi_1(Y_1); \mathbb{Z}) \\
 \cong \uparrow (j_0)_* & & & & (j_1)_* \uparrow \cong \\
 H_3(Y_0; \mathbb{Z}) & \xrightarrow[\cong]{g_*} & & & H_3(Y_1; \mathbb{Z})
 \end{array} \tag{2}$$

We already know that the top part of the diagram (2) commutes. The center part of diagram (2) commutes since $\alpha = ((\iota_1)_*)^{-1} \circ (\iota_0)_*$. The bottom part of diagram (2) commutes again by Lemma 2.3. Finally by design we have $\alpha = f_* = g_*$.

By Proposition 2.2 we know that $(\iota_0)_*([Y_0]) = (\iota_1)_*([Y_1]) \in H_3(W; \mathbb{Z})$. But by the above this implies that $g_*([Y_0]) = [Y_1]$. \square

4. PROOF OF THEOREM 1.2

Proof of Theorem 1.2. Let Y_0 and Y_1 be irreducible closed connected oriented 3-manifolds. We assume that $Y_0 \geq Y_1$ and that $Y_1 \geq Y_0$. We need to show that Y_0 and Y_1 are homeomorphic. If $\pi_1(Y_0)$ and $\pi_1(Y_1)$ are infinite, it follows by a standard argument, using the Sphere Theorem, that Y_1 is aspherical, see [2, p. 48]. Thus we see that the statement follows from Theorem 1.3.

Therefore it suffices to consider the case that $\pi_1(Y_0)$ or $\pi_1(Y_1)$ is finite. We assume that $Y_0 \geq Y_1$ and that $Y_1 \geq Y_0$. We need to show that Y_0 and Y_1 are homeomorphic. By symmetry we can assume that $\pi_1(Y_1)$ is finite.

Let W_0 be a ribbon \mathbb{Q} -homology cobordism from Y_0 to Y_1 and let W_1 be a ribbon \mathbb{Q} -homology cobordism from Y_1 to Y_0 . By Corollary 1.6 we know that the inclusion induced maps $\pi_1(Y_1) \rightarrow \pi_1(W_0)$ and $\pi_1(Y_0) \rightarrow \pi_1(W_1)$ are isomorphisms and by Proposition 2.1 we know that the inclusion induced maps $\pi_1(Y_0) \rightarrow \pi_1(W_0)$ and $\pi_1(Y_1) \rightarrow \pi_1(W_1)$ are monomorphisms.

It follows that $|\pi_1(Y_1)| \leq |\pi_1(W_1)| = |\pi_1(Y_0)| \leq |\pi_1(W_0)| = |\pi_1(Y_1)|$. Since $\pi_1(Y_1)$ is finite we see that we have equalities throughout and we see that all the inclusion induced maps are isomorphisms. In particular we see that $\pi_1(Y_0) \cong \pi_1(Y_1)$.

First we consider the case that $\pi_1(Y_0)$, and thus also $\pi_1(Y_1)$, is not cyclic. In this setting it follows from Theorem 2.4 that Y_0 is homeomorphic to Y_1 .

Finally we consider the case that $\pi_1(Y_0)$, and thus also $\pi_1(Y_1)$, is cyclic. It follows from [2, p. 25] that both Y_0 and Y_1 are lens spaces. But it follows almost immediately from Theorem 2.5 that if for two lens spaces Y_0 and Y_1 we have $Y_1 \geq Y_0$ and if they have isomorphic fundamental groups, then Y_0 and Y_1 are homeomorphic. \square

5. SKETCH OF PROOF OF THEOREM 1.5

In this section we provide a sketch of proof of Theorem 1.5 and Corollary 1.6, both of which are essentially due to Agol [1], although he has formulated it in the context of knot complements.

Sketch of proof of Theorem 1.5. This follows almost verbatim in the same way as in all but the last paragraph of [1, Proof of Theorem 1.2]. The only comment to make is that his proof uses residual finiteness of fundamental groups of knot complements. This is due to Hempel [6], using Thurston's proof of his geometrization conjecture for Haken manifolds. In our situation, we need the fact that all fundamental groups of 3-manifolds are residually finite, and this uses the full geometrization conjecture together with Hempel's result. We will outline this proof for the sake of completeness.

Suppose that W is a rational ribbon homology cobordism from Y_- to Y_+ , where both Y_+ and Y_- are homeomorphic to Y . For a finitely presented group π , Agol considers the representation variety $R_N(\pi) = \text{Hom}(\pi, SO(N))$, for some $N \geq 1$, and in the case of a path-connected topological space X , he defines $R_N(X) := R_N(\pi_1(X))$. This representation variety is a real algebraic set. Since the inclusion $\iota_+ : Y_+ \rightarrow W$ defines a surjection at the level of fundamental groups, we obtain an injection $\iota_+^* : R_N(W) \rightarrow R_N(Y_+)$ by precomposition of representations with $(\iota_+)_* : \pi_1(Y_+) \rightarrow \pi_1(W)$. In fact, since there is a presentation of $\pi_1(W)$ obtained by one from $\pi_1(Y_+)$ by only possibly adding relations, but no generators, one can realize $R_N(W)$ as an algebraic subset of $R_N(Y_+)$, $R_N(W) \subseteq R_N(Y_+)$.

On the other hand, the inclusion $\iota_- : Y_- \rightarrow W$ induces an injection of fundamental groups, and Daemi, Lidman, Vela-Vick, and Wong have shown that the induced map $(\iota_-)^* : R_N(W) \rightarrow R_N(Y_-)$ is surjective, see [3, Proposition 2.1]. Both of these results build on work of Gerstenhaber and Rothaus, the

first statement uses [4, Theorem 2], using the residual finiteness of $\pi_1(Y_-)$, and the second statement builds on [4, Theorem 1], where it is essential that the Lie group that Agol considers, the group $SO(N)$, is compact.

At this stage, $R_N(Y_-)$ and $R_N(Y_+)$ have a priori been considered using different presentations, but a sequence of Tietze moves between the presentations induces a polynomial isomorphism $\phi: R_N(Y_-) \rightarrow R_N(Y_+)$ between these. Together with the statement in the last paragraph, one obtains a surjective polynomial map $\phi \circ (\iota_-)^*: R_N(W) \rightarrow R_N(Y_+)$. At this stage Agol uses the following algebraic geometric lemma ([1, Lemma A.2]): If X and Z are real algebraic sets, with $X \subseteq Z$, and if there is a surjective polynomial map $\varphi: X \rightarrow Z$, then $X = Z$. Applied to our problem, this implies that $R_N(Y_+) = R_N(W)$, induced by the inclusion $(\iota_+)^*: R_N(W) \rightarrow R_N(Y_+)$. Finally, by using residual finiteness of $\pi_1(Y_+)$ again, and by the fact that any finite group embeds into some $SO(N)$ for sufficiently large N , we conclude as in Agol's situation that $(\iota_+)_*: \pi_1(Y_+) \rightarrow \pi_1(W)$ is an isomorphism, using $R_N(Y_+) = R_N(W)$. \square

Proof of Corollary 1.6. Suppose that W_0 is a ribbon \mathbb{Q} -homology cobordism from Y_0 to Y_1 , and that W_1 is a ribbon \mathbb{Q} -homology cobordism from Y_1 to Y_0 . We denote by $\iota_{10}: Y_1 \rightarrow W_0$ the natural inclusion. We form a \mathbb{Q} -homology ribbon cobordism from $Y_- := Y_1$ to $Y_+ := Y_1$ by gluing W_0 and W_1 along Y_0 , and we denote by $\iota_+: Y_+ \rightarrow W$ the natural inclusion. Finally, we denote by $j: W_0 \rightarrow W$ the natural inclusion. Then clearly $\iota_+ = j \circ \iota_{10}$. This induces a commutative diagram between fundamental groups

$$\begin{array}{ccc}
 \pi_1(Y_1) & & \\
 (\iota_{10})_* \downarrow & \searrow^{(\iota_+)_*} & \\
 \pi_1(W_0) & \xrightarrow{j_*} & \pi_1(W).
 \end{array} \tag{3}$$

By Theorem 1.5, the map $(\iota_+)_*$ is an isomorphism. Since W_0 is a ribbon \mathbb{Q} -homology cobordism from Y_0 to Y_1 it follows from Proposition 2.1 (1) that the map $(\iota_{10})_*$ is a surjection. By commutativity of the diagram we conclude that $(\iota_{10})_*$ is a monomorphism. Thus in summary it is in fact an isomorphism. \square

REFERENCES

- [1] Ian Agol, *Ribbon concordance of knots is a partial ordering*, arXiv:2201.03626, 2012.

- [2] Matthias Aschenbrenner, Stefan Friedl, and Henry Wilton, *3-manifold groups*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2015. MR 3444187
- [3] Alikbar Daemi, Tye Lidman, David Shea Vela-Vick, and C.-M. Michael Wong, *Ribbon homology cobordisms*, arXiv:1904.09721, 2019.
- [4] Murray Gerstenhaber and Oscar S. Rothaus, *The solution of sets of equations in groups*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1531–1533. MR 166296
- [5] Cameron McA. Gordon, *Ribbon concordance of knots in the 3-sphere*, Math. Ann. **257** (1981), no. 2, 157–170. MR 634459
- [6] John Hempel, *Residual finiteness for 3-manifolds*, Combinatorial group theory and topology (Alta, Utah, 1984), Ann. of Math. Stud., vol. 111, Princeton Univ. Press, Princeton, NJ, 1987, pp. 379–396. MR 895623
- [7] Marius Huber, *Ribbon cobordisms between lens spaces*, Pac. J. Math. **315** (2021), no. 1, 111–128.
- [8] Peter Orlik, *Seifert manifolds*, Lect. Notes Math., vol. 291, Springer, Cham, 1972.
- [9] Peter Scott, *There are no fake Seifert fibre spaces with infinite π_1* , Ann. Math. (2) **117** (1983), 35–70.
- [10] Friedhelm Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. Math. (2) **87** (1968), 56–88.

STEFAN FRIEDL, UNIVERSITÄT REGENSBURG, 93040 REGENSBURG, GERMANY
Email address: sfriedl@gmail.com

FILIP MISEV, UNIVERSITÄT REGENSBURG, 93040 REGENSBURG, GERMANY
Email address: filip.misev@mathematik.uni-regensburg.de

RAPHAEL ZENTNER, UNIVERSITÄT REGENSBURG, 93040 REGENSBURG, GERMANY
Email address: raphael.zentner@mathematik.uni-regensburg.de