MNS ALGEBRA: PROBLEMS

Field Extensions

1. Find the smallest subfield of $\mathbf{C}$ which contains
(a) 0 and 1; (b) 0; (c) 0, 1 and i; (d) i and $\sqrt{2}$; (e) $\sqrt{2}$ and $\sqrt{3}$.

2. Describe the elements of the field $\mathbf{Q}(\sqrt[3]{5})$ and find $[\,\mathbf{Q}(\sqrt[3]{5}) : \mathbf{Q}\,]$.

3. Describe the elements of the field $\mathbf{Q}(\sqrt[3]{5}, i)$ and find $[\,\mathbf{Q}(\sqrt[3]{5}, i) : \mathbf{Q}\,]$.

4. Is $\{\, a + b\sqrt[3]{2} \mid a, b \in \mathbf{Q}\,\}$ a field?

5. Is $\{\, a + b\sqrt{2} + c\sqrt{3} \mid a, b, c \in \mathbf{Q}\,\}$ a field?

6. Show that the intersection of any (non-empty) collection of fields is itself a field.

7. Find the minimal poynomials for the complex numbers $(\sqrt{5}+1)/2$ and $(i\sqrt{3}-1)/2$ over $\mathbf{Q}$.

8. Supply a polynomial in $\mathbf{Q}[t]$ which has $\sqrt{2}+\sqrt{3}$ as a root.

9. Prove that $\mathbf{Q}(\sqrt{2}, \sqrt{3}) = \mathbf{Q}(\sqrt{2}+\sqrt{3})$.

10. Describe the elements of an extension field $\mathbf{Q}(\alpha)$ over $\mathbf{Q}$ when α has the following minimal polynomial over $\mathbf{Q}$:
(a) $t^2 - 5$, (b) $t^4 + t^3 + t^2 + t + 1$, (c) $t^3 + 2$.

11. Given segments of lengths 1, a and b, with $a > b$ and $b > 0$, show how to construct segments of lengths $a + b$, $a - b$, ab and a/b using ruler and compass.

12. Show that an equilateral triangle can be constructed using ruler and compass.

13. Show how to construct the points trisecting a line segment, and the tangent to a circle at a given point, using ruler and compass.

14. Can the angle $2\pi/5$ be trisected using ruler and compass?

15. Show that the regular 11-gon cannot be constructed using ruler and compass.

16. Show that the regular 48-gon and the regular 30-gon can both be constructed using ruler and compass.