Towards fully eAccessible documents

Sam Fearn and Anne Taormina

Contents

1 Introduction to Fourier series

Fourier analysis is an application of Calculus, which is at the heart of modern mathematics and science. It provides a useful tool when studying problems involving vibrations or oscillations. Some obvious examples are vibrating tuning forks, weights attached to springs, sound waves, water waves and alternating electric currents, but Fourier actually developed this theory while attempting to solve physical problems with extremely important applications in everyday's life. These problems are posed in terms of partial differential equations (PDEs), and you will encounter some in the second year module *Analysis in Many Variables*. Here are a few:

1. Laplace's equation: $\nabla^2 u = 0$, where the function u could be (i) the gravitational potential in a region containing no matter, (ii) the electrostatic potential in a charge-free region, (iii) the steady-state (i.e. time-independent) temperature in a region containing no source of heat, (iv) the velocity potential for an incompressible fluid with no vortices and no sources or sinks.

2. Poisson's equation: $\nabla^2 u = f(x, y, z)$, where $f(x, y, z)$ (source density) is a function describing matter, electric charge, a source of heat or fluid (the left hand side has the same meaning as in Laplace's equation).

3. The diffusion or heat flow equation: $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$, where u can be (i) a non steady-state temperature in a region with no heat sources, (ii) the concentration of a diffusing substance. α^2 is a constant.

4. The wave equation: $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$, where u could be (i) the displacement from equilibrium of a vibrating string, membrane or medium (gas, liquid, solid), (ii) the current or potential along a transmission line, (iii) a component of the electric or magnetic field in an electromagnetic wave (light, radio wave,...). v is the speed of propagation of the wave.

Wave motion can often be decomposed into a combination of harmonic oscillations, and Fourier analysis consists in decomposing a function f defined on an interval $(-p, p)$ into an infinite series of sines and cosines, called the *Fourier series of f*. Apart from solving the PDEs mentioned above, Fourier series are also useful in signal processing. For instance, if one receives a signal in the form of a periodic pulse, it is a superposition of a true signal and some noise, which is usually a high frequency signal. It is important to separate the true signal from the noise. This technique is called '*filtering*', and will be described briefly later on.

2 Preliminaries

2.1 Simple harmonic motion - a lightning refresher

Let P be a particle moving at constant speed around a circle of radius A, and Q be a particle moving up and down the segment tb in such a way that the y-coordinate of P and Q are equal at all times.

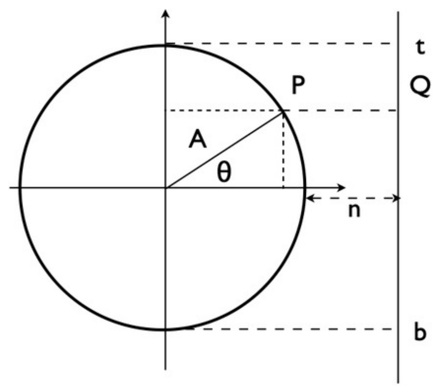


FIGURE 1: *Simple harmonic motion.*

The points P and Q have coordinates

$$P = (A\cos\,\omega t, A\sin\,\omega t), \quad Q = (A + n, A\sin\,\omega t),$$

where ω is the angular velocity of the particle P, with $\omega t = \theta$. The positive real number n is not essential for our purpose.

Simple harmonic motion: an object that moves in such a way that its displacement from equilibrium can be written as $A\sin\,\omega t$ (or $A\cos\,\omega t$, or $A\sin(\omega t+\varphi)$) is said to execute a simple harmonic motion. In our setup, the particle Q executes a simple harmonic motion.

Amplitude of vibration: Maximum displacement (of the object executing the simple harmonic motion) from its equilibrium position. In the situation corresponding to Fig. 1, the amplitude is A.

Period of simple harmonic motion: time needed to complete one oscillation (for the particle P to revolve once around the circle, i.e. time T such that $\theta \equiv 2\pi = \omega T$. In mathematical terms, the period of the function $F(t) = A\sin\omega t$ is the smallest positive real number T such that $F(t+T) = F(t)$. It is indeed $T = \frac{2\pi}{\omega}$ as

$$A \sin(\omega[t + \frac{2\pi}{\omega}]) = A \sin(\omega t + 2\pi) = A \sin \omega t.$$

Frequency of simple harmonic motion: the inverse of the period T, i.e. $f = \frac{1}{T} = \frac{\omega}{2\pi}$.

All these concepts are illustrated in Fig. 2, which graphs the function $A \sin \omega t$ as a function of t. (Note that by choosing the origin of time appropriately, the graph also describes the functions $A \cos \omega t$ and $A \sin(\omega t + \varphi)$).

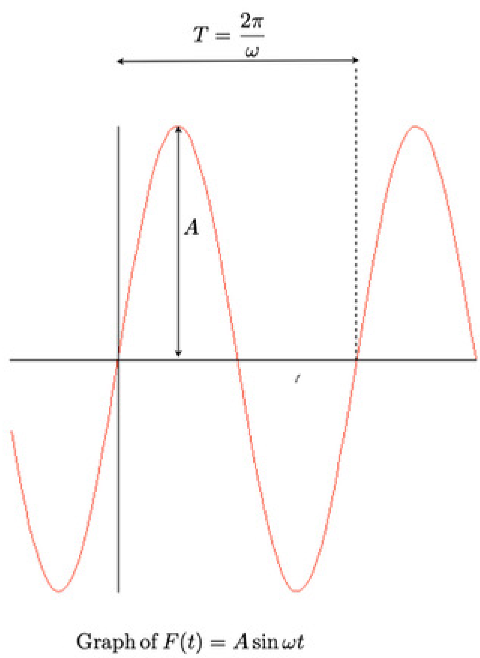


FIGURE 2: *Amplitude and period of* $F(t)$.

2.2 Orthogonal functions

Consider two real-valued functions of a real variable, f_1 and f_2, defined on the interval $[a, b]$. Introduce the inner product of f_1 and f_2 on $[a, b]$ as,

$$(f_1, f_2) \equiv \int_a^b f_1(x)\, f_2(x)\, dx.$$

Example 2.1 Take $f_1(x) = x,\ f_2(x) = x^2,\ [a,b] = [0,1]$, then

$$(f_1, f_2) = \int_0^1 x^3\, dx = \frac{1}{4}[x^4]_0^1 = \frac{1}{4}.$$

Example 2.2 Take $f_1(x) = x^2,\ f_2(x) = x^3,\ [a,b] = [-1,1]$, then

$$(f_1, f_2) = \int_{-1}^1 x^5\, dx = \frac{1}{6}[x^6]_{-1}^1 = 0.$$

In the last example, $(f_1, f_2) = 0$ on $[-1,1]$ and one says that f_1 and f_2 are orthogonal on $[-1,1]$.

More generally, the set

$$\{\cos\frac{\pi k}{p}x\}_{k\in\mathbb{N}} \cup \{\sin\frac{\pi k}{p}x\}_{k\in\mathbb{N}} \tag{2.1}$$

is an orthogonal set on $(-p,p)$, that is, given *non-zero* integer numbers m and n,

$$\int_{-p}^{p} \cos\frac{m\pi}{p}x\ \cos\frac{n\pi}{p}x\, dx = \begin{cases} 0 & \text{for } m,n>0, m\neq n \\ p & \text{for } m=n, \end{cases} \tag{2.2}$$

$$\int_{-p}^{p} \sin\frac{m\pi}{p}x\ \sin\frac{n\pi}{p}x\, dx = \begin{cases} 0 & \text{for } m,n>0, m\neq n \\ p & \text{for } m=n, \end{cases} \tag{2.3}$$

while

$$\int_{-p}^{p} \sin\frac{m\pi}{p}x\ \cos\frac{n\pi}{p}x\, dx = 0 \tag{2.4}$$

for $m,n>0, m\neq n$ as well as for $m=n$. Moreover,

$$\int_{-p}^{p} 1\cdot\cos\frac{m\pi}{p}x\, dx = 0 \qquad \text{for } m>0, \tag{2.5}$$

$$\int_{-p}^{p} 1\cdot\sin\frac{m\pi}{p}x\, dx = 0 \qquad \text{for } m>0. \tag{2.6}$$

These results are proven by explicit integration, using the well-known trigonometric identities

$$\sin(a\pm b) = \sin a\cos b \pm \sin b\cos a,$$

$$\cos(a\pm b) = \cos a\cos b \mp \sin a\sin b.$$

For $p=\pi$, one obtains the following important relations

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin mx\,\cos nx\,dx = 0 \qquad \text{for } m,n\in\mathbb{Z},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin mx\,\sin nx\,dx = \begin{cases} 0 \text{ if } m\neq n, \\ \frac{1}{2} \text{ if } m=n\neq 0, \\ 0 \text{ if } m=n=0, \end{cases}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos mx\,\cos nx\,dx = \begin{cases} 0 \text{ if } m\neq n, \\ \frac{1}{2} \text{ if } m=n\neq 0, \\ 1 \text{ if } m=n=0. \end{cases}$$

3 Fourier Series

3.1 Definition

The aim is to represent functions $f(x)$ of period $2p$ in terms of a sum of the constant function 1 and the trigonometric functions in the set (2.1), which are all of period $2p$. Starting with $f(x)$ defined on $(-p,p)$, the trigonometric series is of the form

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}[\,a_n\cos\frac{n\pi}{p}x+b_n\sin\frac{n\pi}{p}x\,], \tag{3.1}$$

with the coefficients being the constants,

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{3.2}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\cos\frac{n\pi}{p}x\,dx \tag{3.3}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\,\sin\frac{n\pi}{p}x\,dx. \tag{3.4}$$

The above formulas are often called *Euler formulas*. Note that the coefficient of the constant function 1 is labelled $a_0/2$ rather than a_0; this is for convenience so that the formula for a_n reduces to a_0 for $n=0$.

If the coefficients are such that the series (3.1) converges, then its sum will be a function of period $2p$.

Definition 3.1 Suppose that $f(x)$ is a given function of period $2p$, which can be represented by a series of the form (3.1), and that this series converges and that its sum is $f(x)$. Then, one writes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\, a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \,], \tag{3.5}$$

and calls (3.5) the Fourier series of $f(x)$.

In this case, the constants a_0, a_n, b_n for $n > 0$ are called the *Fourier coefficients* of $f(x)$.

3.2 Determination of Fourier coefficients

Assume that the function $f(x)$ is integrable on $(-p, p)$, and that it is equal to its Fourier series, as in (3.5). Also assume that the series (3.5) multiplied by $\cos \frac{m\pi}{p} x$ or $\sin \frac{m\pi}{p} x$ converges (this is to allow term by term integration of the series). The Fourier coefficients a_0, a_n, b_n are determined as follows.

- Multiply (3.5) by the number 1 and integrate both sides between $-p$ and p:

$$\int_{-p}^{p} f(x) \cdot 1 \, dx =$$
$$\int_{-p}^{p} \{ \frac{a_0}{2} \cdot 1 + \sum_{n=1}^{\infty} [\, a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \,] \cdot 1 \} \, dx,$$

which, after use of (2.5) and (2.6), yields

$$\int_{-p}^{p} f(x) \, dx = 2p \frac{a_0}{2} = p \, a_0.$$

- Multiply (3.5) by $\cos \frac{m\pi}{p} x$ where $m \neq 0$, and integrate both sides between $-p$ and p:

$$\int_{-p}^{p} f(x) \cdot \cos \frac{m\pi}{p} x \, dx =$$
$$\int_{-p}^{p} \{ \frac{a_0}{2} \cdot \cos \frac{m\pi}{p} x + \sum_{n=1}^{\infty} [\, a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \,] \cdot \cos \frac{m\pi}{p} x \, \} dx,$$

which gives, after use of (2.5), (2.2) and (2.4),

$$\int_{-p}^{p} f(x) \cos \frac{m\pi}{p} x \, dx = p \, a_m.$$

• Multiply (3.5) by $\sin \frac{m\pi}{p} x$ where $m \neq 0$, and integrate both sides between $-p$ and p:

$$\int_{-p}^{p} f(x) \cdot \sin \frac{m\pi}{p} x \, dx =$$
$$\int_{-p}^{p} \{ \frac{a_0}{2} \cdot \sin \frac{m\pi}{p} x + \sum_{n=1}^{\infty} [\, a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \,] \cdot \sin \frac{m\pi}{p} x \, \} dx,$$

which gives, after use of (2.6), (2.4) and (2.3),

$$\int_{-p}^{p} f(x) \sin \frac{m\pi}{p} x \, dx = p \, b_m.$$

Example 3.2 Expand

$$f(x) = \begin{cases} 0 \text{ for } -\pi < x < 0, \\ \pi - x \text{ for } 0 \leq x < \pi \end{cases} \tag{3.6}$$

in a Fourier series. The graph of $f(x)$ is given in Fig. 3.

Solution: Here, $p = \pi$ and application of (3.2) yields

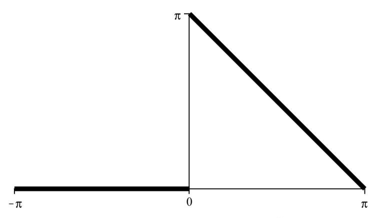


FIGURE 3: *Graph of f on* $(-\pi, \pi)$.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{0} 0 \, dx + \frac{1}{\pi} \int_{0}^{\pi} (\pi - x) \, dx = \frac{\pi}{2}.$$

On the other hand, application of (3.3) yields

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \cos nx \, dx =$$
$$\frac{1}{\pi} \int_{0}^{\pi} (\pi - x) \, \cos nx \, dx = \frac{1}{n\pi} [- \frac{\cos nx}{n}]_0^{\pi} = -\frac{1}{n^2 \pi} [(-1)^n - 1],$$

where integration by parts has been used (set $u = x, dv = \cos nx dx$). Finally, application of (3.4) yields

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,\sin nx\,dx \\ &= \int_0^{\pi}\sin nx\,dx - \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx \\ &= -[\frac{\cos nx}{n}]_0^{\pi} - \frac{1}{\pi}[-\frac{x}{n}\cos nx]_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\,dx \\ &= -\frac{1}{n}[(-1)^n - 1] + \frac{1}{n\pi}[(-1)^n\pi - 0] = \frac{1}{n}, \end{aligned}$$

where integration by parts has been used (set $u = x, dv = \sin nx dx$).

So the Fourier expansion of $f(x)$ on the interval $(-\pi,\pi)$ is given by,

$$\frac{\pi}{4} + \sum_{n=1}^{\infty}\{\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\}. \tag{3.7}$$

One can tidy the final expression a bit more, as $1-(-1)^n = 0$ when n is even, while $1-(-1)^n = 2$ when n is odd. Therefore, only the terms corresponding to n odd will survive in the cosine series of (3.7). Let us thus set $n = 2k-1$, and sum over k rather than n. This yields

$$\frac{\pi}{4} + \sum_{k=1}^{\infty}\{\frac{2}{(2k-1)^2\pi}\cos(2k-1)x + \frac{1}{k}\sin kx\}. \tag{3.8}$$

Example 3.3 The *delta function* $\delta(x)$ is defined to be the 'function' which is equal to 0 everywhere except at $x = 0$, and which satisfies

$$\int_a^b \delta(x)\,f(x)\,dx = f(0)$$

when $0 \in [a,b]$. Note that $\delta(x)$ is not, *actually*, a function. It is a *distribution*, which is a generalization of the function concept, and it is used extensively in mathematical physics. Let us calculate the Fourier coefficients of $\delta(x)$, considered defined on $(-p,p)$. Using the formulas (3.2)-(3.4), we get,

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p}\delta(x)\,dx = \frac{1}{p} \\ a_n &= \frac{1}{p}\int_{-p}^{p}\delta(x)\,\cos\frac{n\pi}{p}x\,dx = \frac{1}{p}\cos 0 = \frac{1}{p} \\ b_n &= \frac{1}{p}\int_{-p}^{p}\delta(x)\,\sin\frac{n\pi}{p}x\,dx = \frac{1}{p}.\sin 0 = 0. \end{aligned}$$

Remark 3.4 The calculation of Fourier coefficients is quite tedious, and if you need to calculate such quantities in your future career, you may benefit from developing a Maple code to this effect. An example of such code is posted on the course webpages (see Handouts). You are encouraged to experiment with it.

3.3 Convergence of a Fourier Series

3.3.1 Sequence of Partial Sums

Fourier series do not always converge, and even if they do converge, they do not necessarily converge to the function that generated them. In order to get insight into convergence, let us consider the sequence of partial sums $\{S_m(x), m \geq 1\}$ of the Fourier series generated by the function $f(x)$, where

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x).$$

If the sequence of partial sums converges to $f(x)$ for some $x \in (-p, p)$, i.e. if

$$\boxed{f(x) = \lim_{m \to \infty} S_m(x)}$$

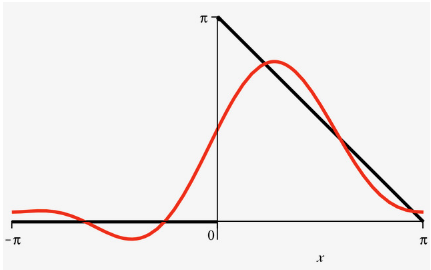
then the Fourier series converges to $f(x)$ at that value of x and one writes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x).$$

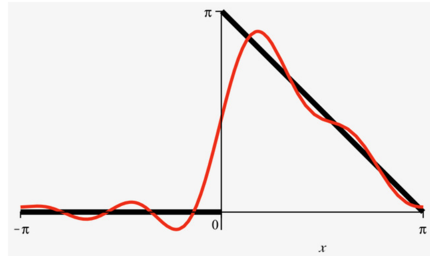
For instance, the m^{th} partial sum corresponding to the Fourier series (3.7) is,

$$S_m(x) = \frac{\pi}{4} + \sum_{n=1}^{m} \{ \frac{1 - (-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \}.$$

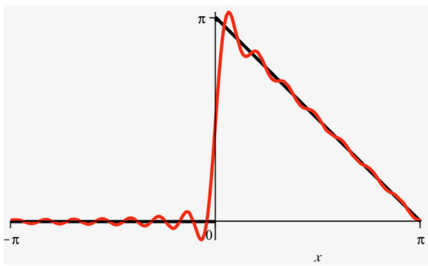
Observe the plots for $S_2(x), S_4(x)$ and $S_{14}(x)$ shown in Fig. 4, and compare with the graph in Fig. 3. As m increases, it becomes more and more difficult to distinguish S_m from the graph of $f(x)$, and we can be fairly confident that the Fourier series of $f(x)$ converges to $f(x)$ for all $x \in (-\pi, \pi)$.



$S_2(x)$



$S_4(x)$



$S_{14}(x)$

FIGURE 4: *Second, fourth and fourteenth partial sums of f.*

3.3.2 Convergence theorem

Theorem 3.5 Let f and f' be continuous functions, except at a finite number of points in the interval $(-p, p)$ (i.e. let f and f' be piecewise continuous on $(-p, p)$), and let them only have finite (jump) discontinuities at these points. Then,

- At a point of continuity x, the Fourier series of f on $(-p, p)$ converges to $f(x)$.
- At a point of discontinuity x_0, the Fourier series converges to the average

$$\frac{f(x_{0+}) + f(x_{0-})}{2},$$

where

$$\begin{aligned} f(x_{0+}) &= \lim_{h\to 0} f(x_0 + h) \qquad h > 0, \\ f(x_{0-}) &= \lim_{h\to 0} f(x_0 - h) \qquad h > 0. \end{aligned}$$

In other words, it converges to the midpoint of the jump. Moreover, if $f(-p) \neq f(p)$, its Fourier series converges to the average of the endpoints at both ends, as illustrated in Fig. 5 (for $p \equiv \pi$).

Proof: Left as an exercise. ■

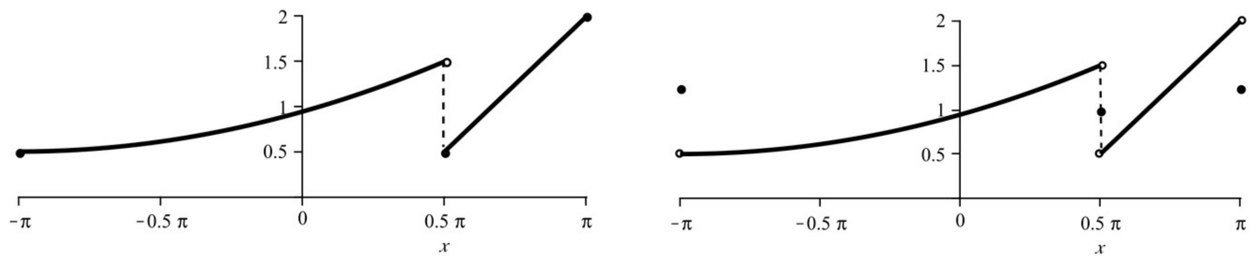


FIGURE 5: *A piecewise continuous function on* $[-\pi, \pi]$ *(left) generates a Fourier series converging to the function on the right.*

In summary,

If the function $\mathsf{f}(\mathsf{x})$ is piecewise smooth on $\mathsf{x} \in (-\mathsf{p}, \mathsf{p})$ then the sequence of truncated Fourier series converges as follows:

$$\lim_{\mathsf{N}\to\infty} \mathsf{S}_\mathsf{N}(\mathsf{x}) \to \frac{1}{2}\left(\lim_{\mathsf{y}\downarrow\mathsf{x}} \mathsf{f}(\mathsf{y}) + \lim_{\mathsf{y}\uparrow\mathsf{x}} \mathsf{f}(\mathsf{y})\right)$$

$$= \begin{cases} \mathsf{f}(\mathsf{x}) \text{ if f is continuous at x,} \\ \text{average of } \mathsf{f}(\mathsf{x}) \text{ across jump at a} \\ \quad \text{discontinuity in } \mathsf{f}(\mathsf{x}) \end{cases}$$

($\mathsf{y} \downarrow \mathsf{x}$ and $\mathsf{y} \uparrow \mathsf{x}$ denote the one-sided limits as y approaches x from above or below respectively.)

Remark 3.6 *The above also illustrates how to produce sans serif fonts for both text and formulas (through textsf and mathsf).*

4 Miscellaneous

A few more environments that Pandoc processes without problems.

The three matrices $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3$ below are rendered reasonably well in html via Pandoc.

$$\mathsf{M}_1 = \begin{bmatrix} \mathsf{a} & \mathsf{b} & \mathsf{c} \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \mathsf{M}_2 = \begin{pmatrix} \mathsf{x}^2 & \frac{\mathsf{a}}{\mathsf{b}} & \mathsf{c}/\mathsf{b} \\ c & d & e \\ f & g & h \end{pmatrix}, \quad M_3 = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$