GMP III - EPIPHANY TERM 2022

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PLAN: (possibly subject to changes)

5. U(1) gauge theory reloaded ← TODAY: recap, conventions, remarks, appetizers ...
6. Non-abelian gauge theories
7. Fibre bundles, connections, and curvature
8. Coupling to charged fields: vector bundles and sections
9. Topological solitons and instantons

(10. Spinors and index theorems)

This is a new module, so I'll write and post lecture notes and exercises as we go along. I will need your help to gauge the level of the course, so please ask questions (during lectures if live, office hours, or on the Discussions board in Ultra) and tell me if you are struggling with some of the content.

Office hours /Q&A for weeks 11,12 (online): Wednesday 12:00. (Zoom link on Ultra)

We'll choose a time for the rest of the term later.

5. U(1) GAUGE THEORY RELOADED

Reminder, conventions, technical and conceptual remarks, slight reformulation useful for next chapter, and appetizers for what's to come.

5.1 - U(1) global symmetry

Consider a complex scalar field ϕ. $\left[\begin{array}{l}\phi : \mathbb{R}^{1,3} \longrightarrow \mathbb{C} \\ \quad x \longmapsto \phi(x)\end{array}\right]$.

The action

$$S_0[\phi,\bar{\Phi}] = \int d^4x \, \mathcal{L}_0(\phi,\bar{\Phi},\partial_\mu\phi,\partial_\mu\bar{\Phi}) \, ,$$

$$\mathcal{L}_0 = -|\partial_\mu\phi|^2 - V(\phi,\bar{\Phi}) = -|\partial_\mu\phi|^2 - U(|\phi|^2) \qquad (5.1)$$

is invariant under global $G = U(1)$ transformations $(g = e^{i\alpha} \in U(1))$

$$g: \quad \phi(x) \longmapsto e^{i\alpha}\phi(x) \qquad \phi \text{ has charge } 1.$$

where $\alpha \sim \alpha + 2\pi$ is a constant parameter.

- REMARKS:

1) $\exists$ conserved current

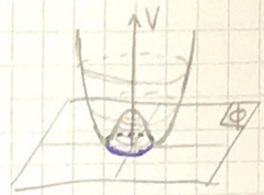
$$j_\mu = i(\bar{\Phi}\,\partial_\mu\phi - \phi\,\partial_\mu\bar{\Phi})$$
$$\partial_\mu j^\mu = 0 \quad \text{(using EoM)} \qquad (5.2)$$

equation of motion (= Euler-Lagrange eqn)

and conserved charge

$$Q = \int d^3x \, j_0$$
$$\frac{d}{dt} Q = 0 \, . \qquad (5.3)$$

2) Global symmetry relates physically distinct configurations.



* Ex 1 : $V = \lambda\left(|\phi|^2 - a^2\right)^2$

$\lambda, a > 0$

1) Calculate the energy/Hamiltonian (Noether charge for $t \mapsto t+c$)

$$E = \int d^3x \left(|\partial_0\phi|^2 + |\partial_i\phi|^2 + V(\phi,\bar\phi)\right)$$

2) Show that the configurations of least energy ("vacua"/"ground states") parametrize a circle.

3) Show that different vacua are related by a global $U(1)$ transformation.

5.2 – U(1) gauge symmetry

Promote the parameter

$$\underset{\text{const.}}{\alpha} \rightsquigarrow \alpha(x)$$

function of spacetime (approaching 0 at infinity!)

The action

$$S[\phi,\bar\phi,A_\mu] = \int d^4x\, \mathcal{L}(\phi,\bar\phi,A_\mu,\partial_\mu\phi,\partial_\mu\bar\phi,\partial_\mu A_\nu),$$

$$\mathcal{L} = \underbrace{\mathcal{L}_0(\phi,\bar\phi,D_\mu\phi,D_\mu\bar\phi)}_{= -\overline{D_\mu\phi}\, D^\mu\phi - U(|\phi|^2)} + \underbrace{\mathcal{L}_{Maxwell}}_{= -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}} \qquad (5.4)$$

where A_μ is a real gauge field (/"connection") and

$$D_\mu\phi := (\partial_\mu - iA_\mu)\phi \qquad \text{"COVARIANT DERIVATIVE" of } \phi$$

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \text{"FIELD STRENGTH" (/"curvature")} \qquad (5.5)$$

is invariant under $G = U(1)$ gauge transformations
/local

$$\phi(x) \mapsto e^{i\alpha(x)} \phi(x)$$
$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \alpha(x) . \tag{5.6}$$

• **REMARKS:**

1) To linear order in A_μ

$$\mathcal{L} = \mathcal{L}_0 - j^\mu A_\mu + \dots \tag{5.7}$$

A common alternative normalization is obtained by rescaling $A_\mu \to g A_\mu$ so that g appears in covariant derivatives and

$$\mathcal{L} = \mathcal{L}_0 - g\, j^\mu A_\mu + \dots$$

The gauge coupling g controls the strength of the coupling between j^μ and A_μ. (We'll typically stick to the normalization w/ $-\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu}$.)

2) The gauge group

$$\mathcal{G} = \mathcal{U}(1) := \left\{ g : \begin{array}{l} \mathbb{R}^{1,3} \longrightarrow G = U(1) \\ x \longmapsto g(x) = e^{i\alpha(x)} \end{array} \right\} \tag{5.8}$$

is ∞-dimensional: independent transfo's at different points.

3) "Gauge symmetry" relates physically equivalent configurations, which are to be identified.

NOT A SYMMETRY, BUT RATHER A REDUNDANCY IN OUR DESCRIPTION.

→ Non-trivial topology ⇒ ∃ solitons.

4) Under a U(1) gauge transfo (5.6) (omit x-dependence from now on)

$$D_\mu \phi \longmapsto e^{i\alpha} D_\mu \phi \qquad \text{"gauge covariant"}$$
$$F_{\mu\nu} \longmapsto F_{\mu\nu} \qquad \text{"gauge invariant"} \tag{5.9}$$

5) It's useful to think of $D_\mu = \partial_\mu - iA_\mu$ as a differential operator, which acts on everything to its right: ∂_μ acts by differentiation, A_μ acts by multiplication. (like all functions of x)

Requiring that under a U(1) gauge transfo

$$D_\mu \longmapsto \underbrace{e^{i\alpha} D_\mu e^{-i\alpha}}_{= D'_\mu \equiv \partial_\mu - iA'_\mu}, \tag{5.10}$$

so that

$$D_\mu \phi \longmapsto e^{i\alpha} D_\mu \cancel{e^{-i\alpha}} \cancel{e^{i\alpha}} \phi = e^{i\alpha} D_\mu \phi, \tag{5.11}$$

implies

$$A_\mu \longmapsto \underbrace{A_\mu + \partial_\mu \alpha}_{= A'_\mu} \tag{5.12}$$

and vice versa.

Proof:

$\Rightarrow$) $D_\mu \equiv \partial_\mu - iA_\mu \longmapsto D'_\mu \equiv \partial_\mu - iA'_\mu = e^{i\alpha}(\partial_\mu - iA_\mu)e^{-i\alpha}$

$$= e^{i\alpha}(\partial_\mu e^{-i\alpha}) + \cancel{e^{i\alpha}} \cancel{e^{-i\alpha}} \partial_\mu - i \cancel{e^{i\alpha}} \cancel{e^{-i\alpha}} A_\mu$$

$$= \partial_\mu - i(A_\mu + \partial_\mu \alpha)$$

$$\Rightarrow A_\mu \longmapsto A'_\mu = A_\mu + \partial_\mu \alpha .$$

$\Leftarrow$) * Exercise (reminder of term 1).

Furthermore

$$[D_\mu, D_\nu] = -iF_{\mu\nu} \qquad (5.13)$$

$\equiv D_\mu D_\nu - D_\nu D_\mu$

so the field strength controls the non-commutativity of covariant derivatives. (More about this later when we study "curvatures")

Proof : * Ex 2

Hint : calculate $[\partial_\mu, \partial_\nu]$, $[\partial_\mu, f]$, $[f, g]$ where f and g are functions of x which act by multiplication. You can assume that these operators act on smooth functions. For instance,

$$[\partial_\mu, f] = X \qquad \text{(equation between operators)}$$

if $\forall \psi \in C^\infty$

$$[\partial_\mu, f]\psi = \partial_\mu(f\psi) - f\partial_\mu\psi = X\psi .$$

6) $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ holds LOCALLY,
i.e. in a patch contractible to a point (Poincaré Lemma ✓)

In the overlap between two patches $U^{(1)}$ and $U^{(2)}$,

$$A^{(1)}_\mu = A^{(2)}_\mu + \partial_\mu \alpha^{(12)}$$

gauge field on $U^{(1)}$ — gauge field on $U^{(2)}$ — gauge transfo (by "transition function" $\alpha^{(12)}$)

E.g. Space-time $\underset{\text{TIME}}{\mathbb{R}_t} \times (\underset{\text{SPACE}}{\mathbb{R}^3} \setminus \underset{\text{point}}{p})$ is contractible to a 2-sphere surrounding p.

S^2

→ magnetic monopoles.
(magnetic charge at point p)

* Ex 3 : work out how previous formulae change if ϕ has charge q rather than charge 1.

5.3 – Gauge symmetry and gauge fixing

[Tong QFT, section 6]

• The equations of motion (EoM, = Euler-Lagrange eqns) of (5.4)

$$\mathcal{L} = -\underbrace{|D_\mu \phi|^2}_{\text{short-hand for } D_\mu\bar{\phi}D^\mu\phi} - \underbrace{U(|\phi|^2)}_{\equiv V(\phi,\bar{\phi})} - \frac{1}{4g^2}\underbrace{F_{\mu\nu}^2}_{F_{\mu\nu}F^{\mu\nu}}$$

"SCALAR ELECTRODYNAMICS"

[NOTE: − sign in front of $|D_\mu\phi|^2$. This is because $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$]

are

$$1) \qquad D_\mu D^\mu \phi = -\frac{\partial V}{\partial \bar{\phi}} \equiv -\phi\, U'(|\phi|^2)$$

(EoM for $\bar{\phi}$ is obtained by cplx conjugation)

$$2) \qquad \partial_\mu F^{\mu\nu} = g^2 J^\nu \qquad (5.14)$$

where

$$J_\mu = i\left(\bar{\Phi} D_\mu \phi - \phi D_\mu \bar{\Phi}\right) = j_\mu + 2A_\mu|\phi|^2$$

is the conserved current of the gauge theory.

Proof: *Ex 5.

• Under a U(1) gauge transfo (5.6) the EoM transform as

$$1) \longmapsto e^{i\alpha}\cdot 1) \qquad \text{gauge covariant}$$
$$2) \longmapsto 2) \qquad \text{gauge invariant} \qquad (5.15)$$

therefore, if (ϕ, A_μ) solves the EoM (5.14), the gauge transformed fields $(\phi' = e^{i\alpha}\phi,\ A'_\mu = A_\mu + \partial_\mu \alpha)$ also solve the EoM (5.14).

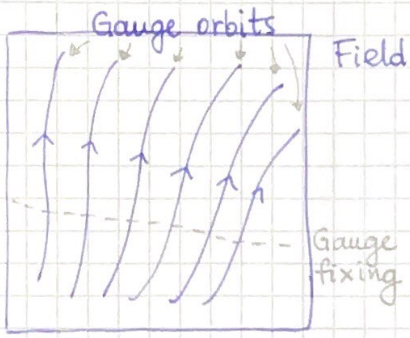
This means that the EoM only determine (ϕ, A_μ) up to a gauge transformation. Given initial data $(\phi^{(0)}, A_\mu^{(0)})$ at a time t_0, we cannot uniquely determine (ϕ, A_μ) at a later time $t > t_0$, because (ϕ', A'_μ) is as good a solution as (ϕ, A_μ), and obeys the same initial condition if $\alpha = 0$ at $t = t_0$.

- We want the EoM to define a well-posed initial value problem and determine uniquely physically observable fields at later times. This requires us to identify

$$(\phi, A_\mu) \sim (\phi' = e^{i\alpha}\phi,\ A'_\mu = A_\mu + \partial_\mu \alpha) \qquad (5.16)$$

Physically observable quantities must be gauge invariant, e.g. $F_{\mu\nu}$, J^μ, $|\phi|^2$ etc.
[This explains remark 3).]

- The picture to keep in mind for gauge theories is



that field space $\mathcal{F}$ is foliated by Gauge orbits

$$\underset{\substack{\shortparallel \\ \mathcal{U}(1)}}{\mathcal{G}} \cdot (\phi(x), A_\mu(x)) = \left\{ \left(e^{i\alpha(x)}\phi(x),\ A_\mu(x) + \partial_\mu \alpha(x)\right) \,\middle|\, \alpha(x) \sim \alpha(x) + 2\pi \right\}$$

In down-to-earth terms, a gauge orbit consists of all field configurations which are related by a gauge transformation.

Physical configuration $\longleftrightarrow$ gauge orbit

[Mathematically, the physical configuration space is the quotient

$$\mathcal{C} = \mathcal{F}/\mathcal{G},$$

namely the set of equivalence classes of field configurations under the equivalence relation (5.16).]

• Rather than working with this redundant description, it is sometimes useful to "fix a gauge" / "pick a gauge", that is, pick a single representative for each gauge orbit.
Any representative is good (they are all physically equivalent), but we need to ensure that we cut each orbit once and only once.

* EXAMPLES:

1) LORENZ GAUGE: impose the constraint

$$\partial_\mu A^\mu = 0, \tag{5.17}$$

(which is Lorentz invariant). This can always be achieved: if we are given a representative A_μ s.t. $\partial_\mu A^\mu \neq 0$, then we can find another representative $A'_\mu = A_\mu + \partial_\mu \alpha$ s.t.

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \alpha = 0 \tag{5.18}$$

by picking α to be a solution of

$$\partial_\mu \partial^\mu \alpha = - \partial_\mu A^\mu . \tag{5.19}$$

Pro: Lorentz invariant.
Con: Incomplete gauge fixing. We are still free to do gauge transformations with $\partial_\mu \partial^\mu \alpha = 0$ and stay in Lorenz gauge.

2) COULOMB GAUGE:

$$\vec{\nabla} \cdot \vec{A} = 0 \tag{5.20}$$

Con: not Lorentz covariant
Pro: the scalar potential A_0 is determined by the charge density $\rho = J_0$

$$A_0(\vec{x}, t) \propto \int d^3x' \, \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|} . \tag{5.21}$$

* **Ex**: determine the proportionality factor.

[Hint: use $\vec{\nabla}^2 \frac{1}{|\vec{x}|} = 4\pi\, \delta^{(3)}(\vec{x})$.]

So if $\rho = 0$, e.g. for pure electromagnetism (no charged "matter" ϕ) we have

$$A_0 = 0$$

in Coulomb gauge.

• **REMARK** (* Ex 6):

We can always (partially) fix a gauge to have $A_0 = 0$.

In this gauge the energy of scalar electrodynamics (5.4) is

$$E = \int d^3x \left(|\partial_0 \phi|^2 + |(\nabla - i\vec{A})\phi|^2 + U(|\phi|^2) + \frac{1}{2g^2}(\vec{E}^2 + \vec{B}^2) \right) \tag{5.22}$$

where

$$E_i = \partial_0 A_i \quad , \quad B_i = -\frac{1}{2}\varepsilon_{ijk} F_{jk} \tag{5.23}$$

are the electric and magnetic field.

5.4 – U(1) Wilson lines and Wilson loops

[Peskin-Schroeder section 15.1]

(Appetizer of chapters 7 & 8)

• <u>Recall</u>: $\partial_\mu \phi$ is not gauge covariant. (ϕ has charge 1)

Indeed the total differential

$$d\phi(x) = \lim_{\varepsilon \to 0} \frac{\phi(x+\varepsilon dx) - \phi(x)}{\varepsilon} = \partial_\mu \phi(x)\, dx^\mu \tag{5.24}$$

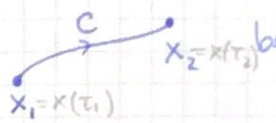
is the difference of 2 terms with different gauge transfo's

$$\phi(x+\varepsilon dx) \longmapsto e^{i\alpha(x+\varepsilon dx)} \phi(x+\varepsilon dx)$$

$$\neq$$

$$\phi(x) \longmapsto e^{i\alpha(x)} \phi(x).$$

This problem can be fixed by introducing the "Wilson line".
(or "parallel transport")

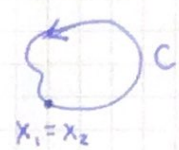
• Let 

be a curve $C : [\tau_1, \tau_2] \longrightarrow \mathbb{R}^{1,3}$

$$\tau \longmapsto x^\mu(\tau)$$

<u>Def</u> – The "<u>Wilson line</u>" (of charge 1) along C is

$$W_C(x_2, x_1) := e^{i\int_{x_1, C}^{x_2} A_\mu(x) dx^\mu} \equiv e^{i\int_{\tau_1}^{\tau_2} d\tau\, A_\mu(x(\tau))\, \dot{x}^\mu(\tau)\, d\tau} \tag{5.25}$$

If C is a closed curve $(x_1 = x_2)$, then 

$$W_C := e^{i\oint_C A_\mu dx^\mu} \tag{5.26}$$

is called a "<u>Wilson loop</u>" of charge 1. (and doesn't depend on $x_1 = x_2$)

• <u>Proposition</u>: under a U(1) gauge transfo (5.6)

$$W_C(x_2, x_1) \longmapsto e^{i\alpha(x_2)} W_C(x_2, x_1)\, e^{-i\alpha(x_1)} \tag{5.27}$$

Proof:

$$W_C(x_2,x_1) = e^{i\int_{x_1,C}^{x_2} A_\mu dx^\mu} \longmapsto e^{i\int_{x_1,C}^{x_2}(A_\mu + \partial_\mu \alpha)dx^\mu}$$

$$= e^{i\int_{x_1,C}^{x_2} A_\mu dx^\mu}\, e^{i\int_{x_1,C}^{x_2} \overbrace{\partial_\mu \alpha\, dx^\mu}^{\text{exact differential}}}$$

$$= W_C(x_2,x_1)\, e^{i[\alpha(x_2)-\alpha(x_1)]}$$

$$= e^{i\alpha(x_2)} W_C(x_2,x_1)\, e^{-i\alpha(x_1)} \qquad \square$$

[Corollary: the U(1) Wilson loop $W_C = e^{i\oint_C A_\mu dx^\mu}$ is gauge invariant.]

Then

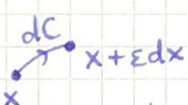
$$W_C(x_2,x_1)\phi(x_1) \longmapsto e^{i\alpha(x_2)} W_C(x_2,x_1)\cancel{e^{-i\alpha(x_1)}}\,\cancel{e^{i\alpha(x_1)}}\phi(x_1) \tag{5.28}$$

$$= e^{i\alpha(x_2)} W_C(x_2,x_1)\phi(x_1)$$

has the same gauge transformation of $\phi(x_2)$!

So it makes sense to consider the covariant differential

$$D\phi^{(x)} \equiv D_\mu \phi^{(x)} dx^\mu = \lim_{\varepsilon\to 0} \frac{\phi(x+\varepsilon dx) - W_{dC}(x+\varepsilon dx, x)\,\phi(x)}{\varepsilon} \tag{5.29}$$

where  is an infinitesimal line element.

Expanding to 1st order

$$W_{dC}(x+\varepsilon dx, x) = e^{i\int_{x,dC}^{x+\varepsilon dx} A_\mu(x')dx'^\mu} \tag{5.30}$$

$$= e^{i\varepsilon A_\mu(x)dx^\mu + O(\varepsilon^2)} = 1 + i\varepsilon A_\mu(x) dx^\mu + O(\varepsilon^2),$$

and substituting in (5.29) we find

$$D\phi(x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\cancel{\phi(x)} + \varepsilon\,\partial_\mu\phi(x)dx^\mu - \cancel{\phi(x)} - i\varepsilon A_\mu(x)\phi(x)dx^\mu + O(\varepsilon^2)\right] \tag{5.31}$$

$$= \left(\partial_\mu\phi(x) - iA_\mu(x)\phi(x)\right)dx^\mu \equiv D_\mu\phi(x)\, dx^\mu$$

which reproduces the previous definition (5.5) of covariant derivative.

(More about this later when we study fibre bundles and sections)

· **REMARKS:**

1) In QM,* The Wilson line $W_c(x_2, x_1)$ is the phase picked up by the wave function of a charged point particle slowly moving from x_1 to x_2 along C in the presence of a gauge field.

2) The Wilson loop $W_C = e^{i\oint_C A_\mu dx^\mu}$ is gauge invariant and therefore physically observable (e.g. Aharonov-Bohm effect in QM). It can happen that

$$\oint_C A_\mu dx^\mu \neq 0$$

even if $F_{\mu\nu} = 0$, if the loop C is not contractible to a point. (More later). 

→ Physical effect associated to A_μ, not to $F_{\mu\nu}$.