6. NON-ABELIAN GAUGE THEORIES

In this chapter we will learn how to formulate gauge theories with a non-abelian gauge group, or Yang-Mills theories, named after Chen Ning Yang and Robert Mills who developed the formalism in 1954.
Non-abelian gauge theories are the language of the Standard Model of Particle Physics, and have led to fruitful interactions between Physics and Maths. (and quite a few Nobel prizes and Fields medals...)

We will spend the rest of the term studying the geometry (and some topology) underlying non-abelian gauge field configurations. But let's introduce the characters first.

6.1 – Compact Lie algebras

[Argyres, section 1.8.1]

(Mostly a review of term 1, with some different conventions.)

- Recall: a Lie algebra $\mathfrak{g}$ is a finite-dimensional vector space, with a further structure provided by the Lie bracket $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

  Basis $\{t_a\}_{a=1}^{\dim \mathfrak{g}}$ $\quad$ t_a : "generators"

$$[t_a, t_b] = i f_{ab}{}^{c} t_c \qquad a,b,c = 1, \dots, \dim \mathfrak{g}. \tag{6.1}$$

where $f_{ab}{}^{c} = -f_{ba}{}^{c}$ are real "structure constants".

- Jacobi identity

$$[[t_a, t_b], t_c] + [[t_b, t_c], t_a] + [[t_c, t_a], t_b] = 0 \tag{6.2}$$

$$\Rightarrow \quad f_{ab}{}^{d} f_{dc}{}^{e} + f_{bc}{}^{d} f_{da}{}^{e} + f_{ca}{}^{d} f_{db}{}^{e} = 0 . \tag{6.3}$$

• An r-dimensional <u>representation</u> (rep) of g is a realization of $\{t_a\}$ as a set of $r \times r$ <u>matrices</u> satisfying (6.1), where now $[\,,\,]$ is the <u>commutator</u> of matrices.

<u>Notation</u>: representation $\underline{r}$ ← [We'll adapt this if there are multiple reps of dimension r]

generators $t_A^{(\underline{r})}$ ← [Omit $(\underline{r})$ if it's clear from the context.]

• A <u>compact</u> <u>Lie algebra</u> is one which can be represented by finite-dimensional <u>Hermitian</u> matrices:

$$t_a^\dagger = t_a \,. \tag{6.4}$$

<u>REMARK</u>:

- The i in (6.1), common in physics literature, is there to ensure (6.4). The maths literature (and Andreas!) uses $\tilde{t}_a = i t_a$, which are anti-hermitian for compact Lie algebras:

$$\tilde{t}_a^\dagger = -\tilde{t}_a \,.$$

• A compact Lie algebra can be decomposed into the direct sum of <u>u(1)</u> Lie algebras and of <u>simple</u> Lie algebras:

$$g = \overbrace{u(1) \oplus \dots \oplus u(1)}^{h} \oplus g_1 \oplus \dots \oplus g_\ell \tag{6.5}$$

$$= \left(\bigoplus_{i=1}^{h} u(1)\right) \oplus \left(\bigoplus_{i=1}^{\ell} g_i\right)$$

1) <u>u(1)</u>: single generator t, so

$$[t, t] = 0 \,. \tag{6.6}$$

Its irreducible representations (irreps) are 1-dimensional:

$$t = q \cdot \mathbb{1} \qquad q \in \mathbb{Z}: \text{"charge"} \tag{6.7}$$

2) simple (compact) Lie algebra:

$$f_{ab}{}^{c} \neq 0 \qquad \forall a \quad (\text{or } b, \text{ or } c) \tag{6.8}$$

* Ex 9.1: show that for a simple Lie algebra

$$\mathrm{tr}_{\underline{r}}(t_a^{(r)}) = 0 . \tag{6.9}$$

There is a basis of generators, which we'll use from now on, s.t.

$$\mathrm{tr}_{\underline{r}}(t_a^{(r)} t_b^{(r)}) = C(\underline{r})\, \delta_{ab} \tag{6.10}$$

where $\mathrm{tr}_{\underline{r}}$ is the trace in the $\underline{r}$ representation (i.e. of $r \times r$ matrices)
$C(\underline{r})$: "quadratic invariant" of $\underline{r}$ $(\neq 0)$

REMARKS:

1) Normalization freedom: we can rescale

$$(t_a, f_{ab}{}^{c}, C(r)) \longrightarrow (\lambda t_a, \lambda f_{ab}{}^{c}, \lambda^2 C(r)) \tag{6.11}$$

leaving previous formulae invariant.

2) For the adjoint rep $\underline{r} = \mathrm{adj}$, (6.10) defines the "Killing form"

$$K: \quad \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto K(v,w) = \mathrm{tr}\left(\mathrm{ad}_v \circ \mathrm{ad}_w\right).$$

where $v = v^a t_a$, $w = w^b t_b$. K is bilinear and symmetric.

3) We can use δ_{ab} (or the Killing form) to lower/raise Lie algebra indices.

* Ex 9.2: show that f_{abc} is totally antisymmetric in a, b, c:

$$f_{abc} = -f_{bac} = -f_{cba} . \tag{6.12}$$

• We can obtain a Lie group G (more precisely: of the connected component of the identity) from a Lie algebra $\mathfrak{g}$ by the exponential map

$$g = e^{i\alpha^a t_a} \qquad (6.13)$$

where $\alpha^a \in \mathbb{R}$.

• In rep $\underline{r}$, g acts as an $r\times r$ unitary matrix:

$$g: \quad \underset{\substack{\text{column}\\ \text{r-dim vector}}}{\phi} \longmapsto \underset{\substack{\text{abstract action}\\ \text{of } G \text{ on } \phi}}{(g\phi)} \equiv \underset{\substack{r\times r\\ \text{matrix}}}{r(g)}\phi = e^{i\alpha^a t_a^{(\underline{r})}}\phi \qquad (6.14)$$

or in components

$$\phi^j \mapsto r(g)^j{}_k\,\phi^k = \left(e^{i\alpha^a t_a^{(\underline{r})}}\right)^j{}_k\,\phi^k . \qquad (6.15)$$

For infinitesimal α, $\phi \mapsto \phi + \delta_\alpha\phi$ with

$$\delta_\alpha \phi^j = i\,\alpha^a \left(t_a^{(\underline{r})}\right)^j{}_k\,\phi^k \qquad (6.16)$$

• The complex conjugate rep. $\overline{\underline{r}}$ has

$$\bar{r}(g) = r(g)^* = \overline{r(g)} \quad \text{cplx conjugate} \qquad (6.17)$$

* Ex 10 :

1) Show that if ϕ transforms in irrep $\underline{r}$, its complex conjugate $\phi^* \equiv \bar{\phi}$ transforms in irrep $\overline{\underline{r}}$.

2) Show that (as $r\times r$ matrices)

$$t_a^{(\bar{r})} = -\left(t_a^{(r)}\right)^T . \qquad (6.18)$$

3) Let $(\phi^j)^* = \bar{\phi}_j$, and construct the row vector $\phi^\dagger = \bar{\phi}^T = (\bar{\phi}_1, \dots, \bar{\phi}_r)$. Show that under the action of G,

$$\phi^\dagger\phi = \bar{\phi}_i\phi^i \longmapsto \phi^\dagger r(g)^{-1} r(g)\phi = \phi^\dagger\phi .$$

• The adjoint representation adj is the irrep given by

$$\left(t_a^{(adj)}\right)^b{}_c = i f_{ac}{}^b \qquad (b,c=1,\dots,\dim g) \qquad (6.19)$$

*Ex 11:

1) Check that (6.19) defines a rep of g.

2) Recall that the adjoint action of the Lie algebra on itself is given by

$$\begin{aligned} ad_x : g &\longrightarrow g \\ y &\longmapsto ad_x(y) := [x,y] \end{aligned}$$

$\forall x \in g$.

Show that

$$ad_{t_a}(y^b t_b) = \left(t_a^{(adj)}\right)^b{}_c \, y^c \, t_b \qquad (6.20)$$

3) Show that the quadratic invariant of the adjoint rep is

$$C(adj) = \frac{f_{abc} f^{abc}}{\dim g} \,, \qquad (6.21)$$

where Lie algebra indices are raised (lowered) using δ^{ab} ($/\delta_{ab}$).

*Ex 12: CLASSICAL COMPACT LIE GROUPS $SU(N)$, $SO(N)$, $USp(2N)$

- $SU(N)$: $g \in Mat_N(\mathbb{C})$ s.t. $g^\dagger g = \mathbb{1}_N$, $\det g = 1$.
- $SO(N)$: $g \in Mat_N(\mathbb{R})$ s.t. $g^T g = \mathbb{1}_N$
- $USp(2N)$: $g \in Mat_{2N}(\mathbb{C})$ s.t. $\begin{cases} g^T J g = J, \\ g^\dagger g = \mathbb{1}_{2N} \end{cases}$ $J = \begin{pmatrix} 0 & \mathbb{1}_N \\ -\mathbb{1}_N & 0 \end{pmatrix}$.

1) Characterize the Lie algebras $su(N)$, $so(N)$, $usp(2N)$ as vector spaces of matrices subject to conditions.

Def - The fundamental (or defining) representation of a matrix group G is the one in which G acts by matrix multiplication:

$$\phi \longmapsto g\cdot\phi \qquad \text{matrix multiplication}$$

that is,

$$r(g)^i{}_j = g^i{}_j \tag{6.22}$$

where g is a matrix.

We denote the fundamental representation by "fund" or $\underline{n}$, where n is its dimension ($=N$ for $SU(N)$, $SO(N)$, $=2N$ for $USp(2N)$).

2) Find the generators of fund and its cplx conjugate rep $\overline{\text{fund}}$ for $SU(N)$, $SO(N)$, $USp(2N)$, and show that fund and $\overline{\text{fund}}$ are equivalent for $SO(N)$ and $USp(2N)$, namely

$$t_a^{(\overline{\text{fund}})} = V\, t_a^{\text{fund}}\, V^{-1}$$

for some invertible matrix V.

(Being the Lie algebras compact, V is unitary: $V^\dagger = V^{-1}$.)

6.2 – Non-abelian gauge theories: the fields

[Argyres 1.8, Tong GFT 2.1]

- This section: introduce the cast of characters that we will then use to formulate actions which are invariant under non-abelian gauge transfo's.

  - Fields ϕ (reps of G)
  - Covariant derivative $D_\mu \phi$
  - Gauge field A_μ
  - Field strength $F_{\mu\nu}$.

  and their gauge transformations.

- We will generalize later, but let's start easy, assuming

  - G : classical group (e.g. $SU(N)$)
  - ϕ : fund rep (" $\underline{N}$)

  i.e., the gauge transfo of ϕ is

$$\phi \longmapsto g\phi = e^{i\alpha^a t_a}\phi \tag{6.23}$$

where ϕ is a N-vector, t_a a $N\times N$ matrix, and g acts on ϕ by matrix multiplication.

Recall: $\phi = \phi(x)$, $g = g(x)$, $\alpha^a = \alpha^a(x)$.

- First, define the covariant derivative

$$D_\mu \phi := \partial_\mu \phi - i A_\mu \phi \tag{6.24}$$

where the gauge field A_μ is now a matrix:

$$A_\mu = A_\mu^a t_a . \tag{6.25}$$

We require that under the non-abelian gauge transfo (6.23)

$$D_\mu \phi \longmapsto g\, D_\mu \phi\,. \tag{6.26}$$

Viewing

$$D_\mu := \partial_\mu \mathbb{1} - i A_\mu \tag{6.27}$$

as a matrix-valued differential operator (we'll typically omit $\mathbb{1}$),

[For mathematically minded students:

$$D_\mu = \partial_\mu \otimes \mathbb{1} - i A_\mu^a(x) \otimes t_a$$

acting on $\underset{\text{patch}}{C^\infty(U)} \otimes \underset{\text{N-dim vector space (fund rep)}}{V}$.]

we require the gauge transfo

$$D_\mu \longmapsto g\, D_\mu\, g^{-1}. \tag{6.28}$$

In terms of the gauge field, this means

$$\begin{aligned} \partial_\mu - iA_\mu \longmapsto \partial_\mu - iA'_\mu &= g(\partial_\mu - iA_\mu)g^{-1} \\ &= g(\partial_\mu g^{-1}) + \cancel{g g^{-1}}\partial_\mu - i g A_\mu g^{-1} \end{aligned} \tag{6.29}$$

NOTE: can't assume that g, A_μ commute!

so the gauge field transforms as (now this is a matrix-valued function, not a diff. op'r)

$$\begin{aligned} A_\mu \longmapsto A'_\mu &= g A_\mu g^{-1} + i g(\partial_\mu g^{-1}) \\ &= g A_\mu g^{-1} - i(\partial_\mu g) g^{-1} \end{aligned} \tag{6.30}$$

where I've used

$$0 = \partial_\mu(\mathbb{1}) = \partial_\mu(g \cdot g^{-1}) = (\partial_\mu g)\cdot g^{-1} + g(\partial_\mu g^{-1}). \tag{6.31}$$

- REMARKS:

- The 1st term in the gauge transfo (6.30) of A_μ is the adjoint action of the Lie group on a Lie algebra element.
- The 2nd term is a correction term.

• Finally, in analogy with the U(1) case we define

$$F_{\mu\nu} := i\,[D_\mu, D_\nu]\,. \tag{6.32}$$

(viewed as a differential operator, which turns out to be multiplicative, that is, it simply acts by multiplication, no differentiations involved.)

By construction, under a gauge transfo (6.23)

$$F_{\mu\nu} \longmapsto g\,F_{\mu\nu}\,g^{-1}\,. \tag{6.33}$$

Proof:

$$\begin{aligned} F_{\mu\nu} = i[D_\mu, D_\nu] \mapsto F'_{\mu\nu} &= i\left[g D_\mu g^{-1}, g D_\nu g^{-1}\right] \\ &= i\,g\,[D_\mu, D_\nu]\,g^{-1} = g F_{\mu\nu} g^{-1}. \qquad \square \end{aligned}$$

Calculating the commutator (6.32), we find that (as a matrix-valued function)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu]\,. \tag{6.34}$$

Proof

$$\begin{aligned} -i\,F_{\mu\nu} = [D_\mu, D_\nu] &= \left[\partial_\mu \mathbb{1} - iA_\mu, \partial_\nu \mathbb{1} - iA_\nu\right] \\ &= [\partial_\mu \mathbb{1}, \partial_\nu \mathbb{1}] - i\,[\partial_\mu \mathbb{1}, A_\nu] - i\,[A_\mu, \partial_\nu \mathbb{1}] - [A_\mu, A_\nu] \\ &= \quad 0 \quad - i\,(\partial_\mu A_\nu)\mathbb{1} + i\,(\partial_\nu A_\mu)\mathbb{1} - [A_\mu, A_\nu] \\ &= -i\,\left(\partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu]\right)\mathbb{1} \qquad \square. \end{aligned}$$

(If you are confused, it might help to restore $\otimes$ and follow the rules for tensor products. Or more easily, simply apply the above to any vector-valued test function ψ in the fund rep.)

• REMARK:

– The finite gauge transfo (6.33) of the field strength $F_{\mu\nu}$ is by the **adjoint action** of the Lie group on the Lie algebra. This means that $F_{\mu\nu}$ transforms in the **adjoint rep** adj.

* **Ex 14**:

By considering infinitesimal gauge transfo's ($|\alpha^a| \ll 1$)

$$g = e^{i\alpha^a t_a} \equiv e^{i\alpha} \simeq 1 + i\alpha + O(\alpha^2) \qquad (6.35)$$

and Taylor expanding to leading order in $\underset{\mathfrak{g} = \mathrm{Lie}(G)}{\alpha}$, show that the infinitesimal gauge variations of the fields are

$$\begin{aligned} &\delta_\alpha \phi = i\alpha\phi && \leftarrow \text{fund rep of } \mathfrak{g} \\ &\delta_\alpha A_\mu = i[\alpha, A_\mu] + \partial_\mu \alpha && \leftarrow \text{"adj rep" + correction term} \\ \Rightarrow\ &\delta_\alpha F_{\mu\nu} = i[\alpha, F_{\mu\nu}]. && \leftarrow \text{adj rep of } \mathfrak{g} \end{aligned} \qquad (6.36)$$

- REMARKS:

1) $F_{\mu\nu}$ transforms in the adj rep of $\mathfrak{g}$ under infinitesimal gauge transfos.
2) A_μ doesn't quite transform in adj, because of the derivative term that we have already encountered for $\mathfrak{g} = u(1)$. (People often say that A_μ transforms in adj, but that's an abuse of terminology.)
3) D_μ does transform in adj.

- All of this generalizes to ϕ an r-vector transforming in an r-dimensional rep $\underline{r}$. One simply needs to replace g in previous formulae by the representation matrix

$$r(g) = e^{i\alpha^a t_a^{(r)}}. \qquad (6.37)$$

For instance

$$D_\mu \phi := (\partial_\mu - iA_\mu)\phi = (\partial_\mu \mathbb{1}_r - iA_\mu^a t_a^{(r)})\phi, \qquad (6.38)$$

and

$$\begin{aligned}F_{\mu\nu}\phi &:= i[D_\mu, D_\nu]\phi \\ &= i\left(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\right)\phi \\ &= i\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a A^b_\mu A^c_\nu\right) t^{(r)}_a \phi\end{aligned} \tag{6.39}$$

where it is understood that if $\phi \in \underline{r}$, then

$$\begin{aligned}A_\mu \phi &:= A^a_\mu t^{(r)}_a \phi , \\ F_{\mu\nu}\phi &:= F^a_{\mu\nu} t^{(r)}_a \phi\end{aligned} \tag{6.40}$$

etc. Similarly, it is customary to simply write $g\phi$ for the abstract action of g on ϕ, which in practice is the multiplication by the representation matrix $r(g)$ if ϕ transforms in $\underline{r}$.

In components,

$$(A_\mu\phi)^i = A^\mu_a \left(t^{(r)}_a\right)^i{}_j \phi^j \qquad (i,j = 1,\dots,r) \tag{6.41}$$

etc.

* Ex 15 :

Show that, if $G = U(1)$, the previous formulae reduce to those introduced in chapter 5 (both for charge 1 and charge q).

* Ex 16 :

Consider ϕ in the adj rep, with components ϕ^a, $a = 1,\dots,\dim g$.

1) Show that

$$(A_\mu\phi)^a = i f_{bc}{}^a A^b_\mu \phi^c \tag{6.42}$$

and similarly for $F_{\mu\nu}\phi$.

2) Let $\Phi := \phi^a t_a$, $A_\mu = A^a_\mu t_a$, $F_{\mu\nu} = F^a_{\mu\nu} t_a$. Show that

$$(A_\mu\phi)^a t_a = [A_\mu, \Phi] \tag{6.43}$$

and similarly for $(F_{\mu\nu}\Phi)$. Show that therefore

$$\begin{aligned} D_\mu \Phi &= \partial_\mu \Phi - i[A_\mu, \Phi] \\ [D_\mu, D_\nu]\Phi &= -i[F_{\mu\nu}, \Phi] \, . \end{aligned} \tag{6.44}$$

and similarly for $(F_{\mu\nu}\Phi)$. Show that therefore

$$\begin{aligned} D_\mu \Phi &= \partial_\mu \Phi - i[A_\mu, \Phi] \\ [D_\mu, D_\nu]\Phi &= -i[F_{\mu\nu}, \Phi] \end{aligned} \qquad (6.44)$$

$\boxed{\text{6.3 - Non-abelian gauge theories: action and EoM}}$

[Argyres 1.8.1
Tong GFT 2.1]

- Start by constructing a gauge invariant action for the (Lie algebra valued) non-abelian gauge field $A_\mu = A_\mu^a t_a$.

Since the field strength $F_{\mu\nu} = F_{\mu\nu}^a t_a$ transforms as

$$F_{\mu\nu} \longmapsto g F_{\mu\nu} g^{-1}, \qquad (6.45)$$

we have that $\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$ is gauge invariant.

Proof

$$\begin{aligned} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) &\longmapsto \mathrm{tr}(g F_{\mu\nu} g^{-1} g F^{\mu\nu} g^{-1}) \\ &\underset{\text{(ciclicity)}}{=} \mathrm{tr}(g^{-1} g F_{\mu\nu} g^{-1} g F^{\mu\nu}) = \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}). \qquad \square \end{aligned}$$

REMARKS:

1) In the previous equations tr is a trace over the vector space on which g naturally acts by matrix multiplication. So here $\mathrm{tr} \equiv \mathrm{tr}_{\underline{\text{fund}}}$ ← trace in fund representation.

2) We could equally well use any other irrep $\underline{r}$, in which case $F_{\mu\nu}^{(r)} = F_{\mu\nu}^a t_a^{(r)}$ and

$$F_{\mu\nu}^{(r)} \longmapsto r(g) F_{\mu\nu}^{(r)} r(g)^{-1}. \qquad (6.46)$$

Then $\mathrm{tr}_{\underline{r}}(F_{\mu\nu}F^{\mu\nu})$ is gauge invariant, by the same logic.

3) Any two such choices are proportional to one another:

$$\mathrm{tr}_{\underline{r}}\left(F^{(\underline{r})}_{\mu\nu}F^{(\underline{r})\mu\nu}\right) = F^{a}_{\mu\nu}F^{b,\mu\nu}\,\mathrm{tr}_{\underline{r}}\left(t^{(\underline{r})}_{a}t^{(\underline{r})}_{b}\right)$$

$$\underset{(6.10)}{=} F^{a}_{\mu\nu}F^{b,\mu\nu}\,C(\underline{r})\,\delta_{ab} \tag{6.47}$$

$$= \underbrace{C(\underline{r})}_{\text{Quadratic invariant of irrep } \underline{r}} F^{a}_{\mu\nu}F^{a,\mu\nu}\,.$$

Then

$$\frac{1}{C(\underline{r})}\,\mathrm{tr}_{\underline{r}}\left(F^{(\underline{r})}_{\mu\nu}F^{(\underline{r})\mu\nu}\right) = F^{a}_{\mu\nu}F^{a,\mu\nu} \tag{6.48}$$

is independent of the choice of representation $\underline{r}$, and is invariant under a change of normalization (6.11).

- The <u>YANG-MILLS ACTION</u> for A_μ is

$$S_{YM}[A] = \int d^4x\,\mathcal{L}_{YM}\,, \qquad \mathcal{L}_{YM} = -\frac{1}{2g^2_{YM}}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right), \tag{6.49}$$

where $\mathrm{tr} \equiv \mathrm{tr}_{\underline{\mathrm{fund}}}$ and we work in a <u>normalization</u> where

$$C(\underline{\mathrm{fund}}) = \frac{1}{2}\,, \tag{6.50}$$

so

$$\mathcal{L}_{YM} = -\frac{1}{4g^2}\,\frac{1}{C(\underline{\mathrm{fund}})}\,\mathrm{tr}_{\underline{\mathrm{fund}}}\left(F_{\mu\nu}F^{\mu\nu}\right)$$

$$= -\frac{1}{4g^2}\,F^{a}_{\mu\nu}F^{a,\mu\nu}\,. \tag{6.51}$$

* Ex 18 :

Show that, for any irrep $\underline{r}$,

$$\mathcal{L}_{YM} = -\frac{1}{2g^2}\frac{1}{T(r)}\,\mathrm{tr}_{\underline{r}}\left(F_{\mu\nu}F^{\mu\nu}\right), \tag{6.52}$$

where the "Dynkin index"

$$T(\underline{r}) := \frac{C(\underline{r})}{C(\underline{\text{fund}})} \tag{6.53}$$

of irrep $\underline{r}$ is invariant under changes of normalization (6.11).

- There is a 2^{nd} gauge invariant term that one can add to the action. It's the "THETA TERM"

$$S_\theta[A] = \int d^4x\, \mathcal{L}_\theta,$$
$$\mathcal{L}_\theta = \frac{\theta}{16\pi^2}\,\mathrm{tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \tag{6.54}$$

where θ is called the "THETA ANGLE" (more about why θ is an angle in later chapters), and

$$\tilde{F}^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{6.55}$$

is the "DUAL FIELD STRENGTH".

- So the general gauge invariant action which includes a kinetic term for A_μ, as well as interaction terms, is

$$S_{gauge}[A] = S_{YM}[A] + S_\theta[A]$$
$$\mathcal{L}_{gauge} = \mathcal{L}_{YM} + \mathcal{L}_\theta = -\frac{1}{2g_{YM}^2}\,\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{\theta}{16\pi^2}\,\mathrm{tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right). \tag{6.56}$$

* Ex 19 :

1) Express the Lagrangian $\mathcal{L}_{gauge}$ in terms of A_μ^a and the structure constants $f_{ab}{}^c$.

2) Show that the theta term can be written as a surface term: /boundary

$$S_\theta = \frac{\theta}{8\pi^2}\int d^4x\, \partial_\mu K^\mu , \qquad \Rightarrow S_\theta \text{ doesn't affect EoM !} \tag{6.57}$$
$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}\left(A_\nu \partial_\rho A_\sigma - \frac{2i}{3} A_\nu A_\rho A_\sigma\right)$$

4) Show that the EQUATIONS OF MOTION obtained from the action S_{gauge} are

$$D_\mu F^{\mu\nu} := \partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}] = 0 . \tag{6.58}$$

5) Show, without using the EoM, that the BIANCHI IDENTITY

$$D_\mu \tilde{F}^{\mu\nu} = 0 \tag{6.59}$$

holds.

• If there are CHARGED FIELDS ϕ transforming in rep. $\underline{r}$, we can write a gauge invariant ~~kinetic term~~ action for them using covariant derivatives. E.g. for $G = SU(N)$

$$S_{matter}[\phi, \phi^\dagger, A_\mu] = \int d^4x\, \mathcal{L}_{matter} ,$$
$$\mathcal{L}_{matter} = -(D_\mu\phi)^\dagger D^\mu\phi - V(\phi, \phi^\dagger) , \tag{6.60}$$

where we require the scalar potential V to be gauge invariant, that is $V \mapsto V$ under gauge transformations.
This generalizes to other classical groups by using the appropriate inner product in the kinetic term.

* Ex 20 :

Consider the action

$$S[\phi,\phi^\dagger,A_\mu] = S_{YM}[A_\mu] + S_\theta[A_\mu] + S_{matter}[\phi,\phi^\dagger,A_\mu] .$$

1) Show that the EoM are

$$D_\mu D^\mu \phi = -\frac{\partial V}{\partial \phi^\dagger} \qquad (6.61)$$

$$D_\mu F^{\mu\nu} = g_{YM}^2 J^\nu$$

for a current $J_\mu = J_\mu^a t_a$ that you should find.

2) Show that under a gauge transformation

$$J^\mu \longrightarrow g J^\mu g^{-1} \qquad (6.62)$$

and that J^μ is "covariantly conserved", namely

$$D_\mu J^\mu = 0 . \qquad (6.63)$$

6.4 – Non-abelian Wilson line and Wilson loop

* ← ADVANCED BONUS MATERIAL (NOT EXAMINABLE)

- The logic will be the same as for $G=U(1)$: we use the Wilson line $W_C(x_2,x_1)$ to "parallel transport" a charged field $\phi(x_1)$ (in QM: the wavefunction $\psi(x_1)$ of a charged particle) from x_1 to x_2 along curve C.

There will be a few technical differences due to the non-abelian (= non-commutative) nature of the gauge field A_μ.

In this section A_μ is viewed as a fixed background. We are interested in how it affects a charged probe $\phi(x)$ (or $\psi(x)$).

• Let's go backwards this time, and define an infinitesimal Wilson line $W_{dC}(x+\varepsilon dx, x)$ from the covariant derivative:

$$D\phi(x) \equiv D_\mu \phi(x)\, dx = \lim_{\varepsilon\to 0} \frac{\phi(x+\varepsilon dx) - W_{dC}(x+\varepsilon dx, x)\,\phi(x)}{\varepsilon} \qquad (6.64)$$

and identifying with

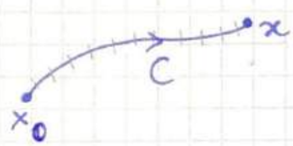
$$D_\mu \phi(x) = \partial_\mu \phi(x) - i A_\mu^{(\underline{r})}(x)\,\phi(x) \qquad (6.65)$$

where $A_\mu^{(\underline{r})}(x) = A_\mu^a(x)\, t_a^{(\underline{r})}$ if ϕ transforms in rep $\underline{r}$, we find

$$W_{dC}(x+dx, x) = \mathbb{1} + i A_\mu^{(\underline{r})}(x)\, dx^\mu + O((dx)^2). \qquad (6.66)$$

(Now we view dx as infinitesimal, and drop ε.)

• Now we'd like to get a finite Wilson line along a finite curve C, by iterating infinitesimal paths dC.



We want to find the line operator $W_C(x, x_0)$ which obeys:

$$\begin{aligned} \partial_\mu W_C(x,x_0) &= i A_\mu^{(\underline{r})}(x)\, W_C(x,x_0) \quad \text{from (6,66)} \\ W_{\underset{\text{C=trivial curve}}{}}(x_0,x_0) &= \mathbb{1} \quad \text{initial cond} \end{aligned} \qquad (6.67)$$

The complication is that the values of A_μ at different points along the curve C don't commute.

Eqn (6.67) can be integrated – I'll spare you the proof – to
but ask me if you are interested

$$\begin{aligned} W_C(x,x_0) &= \mathcal{P} \exp\left(i \int_{x_0, C}^{x} dx'^\mu A_\mu(x') \right) \\ &\equiv \mathcal{P} \exp\left(i \int_{\tau_0}^{\tau} d\tau'\, \dot{x}^\mu(\tau')\, A_\mu(x(\tau')) \right), \end{aligned} \qquad (6.68)$$

where

$$C: [\tau_0, \tau] \longrightarrow \mathbb{R}^{1,3}$$
$$\tau' \longmapsto x^\mu(\tau') \tag{6.69}$$

is a parametrization of the curve C, and $\mathcal{P}\exp(\dots)$ is the "path-ordered exponential". This means that when we Taylor expand the exponential, we order the matrices $A_\mu(x(\tau'))$ such that the values of τ' increases from right to left.

In practice, letting

$$\mathcal{A}(\tau) := \dot{x}^\mu(\tau) A_\mu(x(\tau)), \tag{6.70}$$

one has

NOTE: $\tau_1 \geq \tau_2 \geq \dots \geq \tau_n$

$$\begin{aligned} W_C(x, x_0) &= \sum_{n=0}^{\infty} i^n \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \dots \int_{\tau_0}^{\tau_{n-1}} d\tau_n \, \mathcal{A}(\tau_1)\mathcal{A}(\tau_2)\dots\mathcal{A}(\tau_n) \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau} d\tau_2 \dots \int_{\tau_0}^{\tau} d\tau_n \, \mathcal{T}\left[\mathcal{A}(\tau_1)\mathcal{A}(\tau_2)\dots\mathcal{A}(\tau_n)\right] \end{aligned} \tag{6.71}$$

where the path ordering $\mathcal{P}$ has become a time ordering $\mathcal{T}$ for the affine parameter/"time" τ along the curve:

$$\mathcal{T}\left[f(\tau_1) f(\tau_2) \dots f(\tau_n)\right] = \sum_{\sigma \in S_n} \theta(\tau_{\sigma(1)}, \dots, \tau_{\sigma(n)}) f(\tau_{\sigma(1)}, \dots, \tau_{\sigma(n)})$$

and

$$\theta(\tau_1, \tau_2, \dots, \tau_n) := \begin{cases} 1 & \text{if } \tau_1 \underset{(-)}{>} \tau_2 \underset{(-)}{>} \dots \underset{(-)}{>} \tau_n \\ 0 & \text{else} \end{cases} \tag{6.72}$$

• REMARKS:

1) This is a similar construction to the time evolution operator in QM.

2) Since $A_\mu^{(r)} = A_\mu^a t_a^{(r)}$, we call (6.68) the WILSON LINE

in representation $\underline{r}$, along curve C from x_0 to x.

3) It can be shown that under a gauge transfo, the Wilson line transforms as

$$W_C(x, x_0) \mapsto g(x)\, W_C(x, x_0)\, g^{-1}(x_0) \qquad (6.73)$$

for the WL in the fund. rep, and

$$W_C^{(\underline{r})}(x, x_0) \mapsto r(g(x))\, W_C^{(\underline{r})}(x, x_0)\, r(g^{-1}(x_0)) \qquad (6.74)$$

for a general rep. $\underline{r}$.

So, if

$$\phi(x_0) \longmapsto r(g(x_0))\, \phi(x_0)\,, \qquad (6.75)$$

then

$$W_C^{(\underline{r})}(x, x_0)\, \phi(x_0) \mapsto r(g(x)) \left[W_C^{(\underline{r})}(x, x_0)\, \phi(x_0) \right], \qquad (6.76)$$

namely $W_C^{(\underline{r})}(x, x_0)\phi(x_0)$ has the same **gauge transformation** as $\phi(x)$.

4) If C is a closed curve, then the "WILSON LOOP"

$$\mathcal{W}_C^{(\underline{r})} := \mathrm{tr}_{\underline{r}} \left(P \exp \left(i \oint_C A_\mu^{(\underline{r})}(x)\, dx^\mu \right) \right) \qquad (6.77)$$

is gauge invariant.

NOTE: the Wilson loop depends on the rep $\underline{r}$ and on the curve C.