7. BUNDLES, CONNECTIONS, CURVATURE & SECTIONS

[Ooguri, lectures 2,5;
Nakahara for more details]

- So far: how to formulate gauge theories.

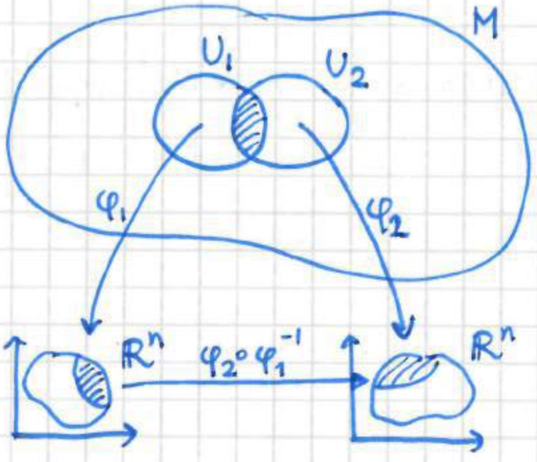
  Next: understand the underlying geometric structure.

[We'll need to learn about fibre bundes, which are a consistent way of adding extra structure on top of a differentiable manifold. The general formal definition is quite abstract, so let's build towards it slowly, by successive generalizations.]

Final goal: describe gauge transformations, gauge fields, field strengths and charged fields ~~mathematically~~ geometrically.

7.1 – The tangent bundle

- Recall: differentiable manifold M (of dimension n)



$\varphi_i : U_i \longrightarrow \mathbb{R}^n$ invertible

(U_i, φ_i): coordinate chart / patch

$\{(U_i, \varphi_i)_{i=I}\}$: atlas

$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$
transition function
(smooth, invertible)

$\varphi_i(p) = (x^1_{(i)}, \dots, x^n_{(i)})$ coordinates of point p (in patch U_i).

We'll often drop (i) unless it's useful.

[Much of the abstraction in the subject is to do with defining notions intrinsically, that is without making reference to coordinates. Things become clearer when you describe them in local coordinates.]

A function

$$\hat{f} : M \longrightarrow \mathbb{R}$$
$$p \longmapsto \hat{f}(p) \qquad (7.1)$$

is differentiable (/smooth) if $\forall\, (U_i, \varphi_i)$, its expression in local coordinates

$$f_{(i)} := \hat{f} \circ \varphi_i^{-1} : \varphi_i(U_i) \longrightarrow \mathbb{R}$$
$$x = (x^1, \ldots, x^n) \longmapsto f_{(i)}(x) \qquad (7.2)$$

is differentiable (/smooth).

$C^\infty(M)$: smooth fns on M.

This extends to other types of maps in the obvious way.

[Later on, when we know what we are doing, we'll abuse notation and won't distinguish the "intrinsic" $\hat{f}(p)$ from its "extrinsic" description $f(x) = (\hat{f} \circ \varphi^{-1})(x)$ in terms of local coordinates $x = \varphi(p)$ in a patch (U, φ). For now, to avoid confusion I'll use hats for the intrinsic objects, and no hats for their expressions in coord's.]

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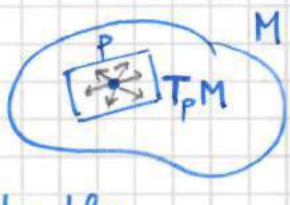
- 1st term: defined
  - tangent vectors to a curve C at $p \in M$
  - tangent space of M at p: T_pM.



Next: extend this construction from a point p to the whole manifold M. Informally, we'd like to define

$$TM = \bigcup_{p \in M} T_pM, \qquad (7.3)$$

a "bundle" of the tangent spaces at different points.

$\rightarrow$ tangent bundle TM of M.

Question: how to define it properly?

• To gain intuition, it's useful to take an equivalent but complementary view of tangent vectors.

Def – A (TANGENT) VECTOR FIELD $\hat{v}$ on M is a map

$$\begin{aligned} \hat{v}: C^\infty(M) &\longrightarrow C^\infty(M) \\ \hat{f} &\longmapsto \hat{v}(\hat{f}) \end{aligned} \tag{7.4}$$

which obeys:

or a (mathematical) field $\mathbb{F}$ more generally

1) LINEARITY: $\forall a_1, a_2 \in \mathbb{R}$, $\forall \hat{f}_1, \hat{f}_2 \in C^\infty(M)$

$$\hat{v}(a_1 \hat{f}_1 + a_2 \hat{f}_2) = a_1 \hat{v}(\hat{f}_1) + a_2 \hat{v}(\hat{f}_2) \tag{7.5}$$

2) LEIBNIZ RULE: $\forall \hat{f}, \hat{g} \in C^\infty(M)$

$$\hat{v}(\hat{f} \cdot \hat{g}) = \hat{v}(\hat{f}) \cdot \hat{g} + \hat{f} \cdot \hat{v}(\hat{g}). \tag{7.6}$$

(like first derivatives! More about this soon when we use coordinates.)

* Ex 21: let $\hat{v}, \hat{w}$ be tangent vector fields.

1) Show that $\hat{w} \circ \hat{v}$ does not define a tangent vector field.

2) " " $[\hat{w}, \hat{v}] = \hat{w} \circ \hat{v} - \hat{v} \circ \hat{w}$ defines a tangent vector field.

Given a tangent vector field $\hat{v}$ on M and a point $p \in M$, we can define a TANGENT VECTOR $\hat{v}_p \in T_pM$ at point p by evaluating everything at point p:

$$\begin{aligned} \hat{v}_p: C^\infty(M) &\longrightarrow \mathbb{R} \\ \hat{f} &\longmapsto \hat{v}_p(\hat{f}) = (\hat{v}(\hat{f}))(p) \end{aligned} \tag{7.7}$$

This obeys

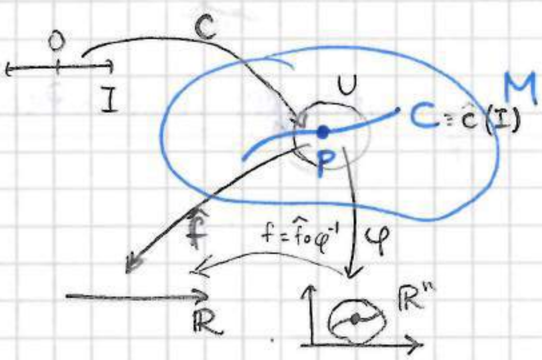
1) $\hat{v}_p(a_1 \hat{f}_1 + a_2 \hat{f}_2) = a_1 \hat{v}_p(\hat{f}_1) + a_2 \hat{v}_p(\hat{f}_2)$

2) $\hat{v}_p(\hat{f} \cdot \hat{g}) = \hat{v}_p(\hat{f}) \cdot \hat{g}(p) + \hat{f}(p) \cdot v_p(\hat{g})$

One can define tangent vectors at a point using this axioms and making no reference to tangent vector fields

- How is this def of tangent vectors related to the previous one (1st term) in terms of curves?

Given a smooth curve



$$\begin{aligned} c:\ I \subseteq \mathbb{R} &\longrightarrow M \\ \tau &\longmapsto c(\tau) \end{aligned} \tag{7.8}$$

with $c(0) = p$, define a tangent vector $\hat{v}_p$ to the curve at p by

$$\hat{v}_p(\hat{f}) := \frac{d}{d\tau}\hat{f}(c(\tau))\Big|_{\tau=0} , \tag{7.9}$$

which is defined intrinsically $\forall\, \hat{f} \in C^\infty(M)$.

Let's express this in coordinates x^μ in a patch (U,φ), where the curve is parametrised by

$$(\varphi \circ c)(\tau) \equiv x(\tau) = (x^1(\tau), \dots, x^n(\tau)) . \tag{7.10}$$

and the function $\hat{f}$ is represented as $f = \hat{f}\circ\varphi^{-1}$ (function of x)

$$\begin{aligned} \hat{v}_p(\hat{f}) &= \frac{d}{d\tau}(\hat{f}\circ c)(\tau)\Big|_{\tau=0} = \frac{d}{d\tau}\big(\underbrace{\hat{f}\circ\varphi^{-1}}_{f}\circ\underbrace{\varphi\circ c}_{x}\big)(\tau)\Big|_{\tau=0} \\ &= \frac{d}{d\tau} f(x(\tau))\Big|_{\tau=0} = \underline{\underline{\dot{x}^\mu(0)}}\frac{\partial}{\partial x^\mu} f(x)\Big|_{x=x(0)=\varphi(p)} \end{aligned} \tag{7.11}$$

(chain rule)

DIRECTIONAL DERIVATIVE of f along the tangent to the curve at point p with coordinates $x = x(0)$

- REMARKS:

1) When you described the tangent vector to a curve at p in local coordinates in 1st term, $\underline{\underline{\dot{x}^\mu(0)}}$ were the components of the tangent vector.

2) To construct a basis of the tangent space T_pM, you used curves C_a s.t. $\dot{x}^\mu(0) = \delta^\mu_a$ $\quad (\vec{\dot{x}}(0) = (0,0,\dots,0,\underset{a}{1},0,\dots,0))$. $\leftarrow x^\mu(\tau) = x^\mu(0) + \tau\delta^\mu_a$
Calling the corresponding tangent vector e_a, we have

$$e_a(\hat{f}) = \frac{\partial}{\partial x^a} f(x)\Big|_{x=\varphi(p)} = \frac{\partial}{\partial x^a} \hat{f}(\varphi^{-1}(x))\Big|_{x=\varphi(p)}, \qquad (7.12)$$

or for short

$$e_a = (\partial_a)_p ,$$

where $(\partial_a)_p$ is $\frac{\partial}{\partial x^a}$ when we work in coord's $x = \varphi(p)$.

- In summary, we can write any tangent vector $\hat{v}_p \in T_pM$ intrinsically as

$$\hat{v}_p = \hat{v}^a (\partial_a)_p , \qquad (7.14)$$

or extrinsically (in local coord's) as

$$v = v^a \frac{\partial}{\partial x^a} , \qquad (7.15)$$

where here $\hat{v}^a = v^a$ $(a=1,\dots,n)$ are real numbers. (the components of the vector)

- Now let's consider a collection of tangent spaces over every point on M: the "<u>TANGENT BUNDLE</u>"

$$TM = \bigcup_{p\in M} T_pM . \qquad (7.16)$$

This is naturally a <u>manifold</u>: for each coordinate chart (U_i, φ_i) on M, we can define coordinates (x^μ, v^μ) on $\bigcup_{p\in U_i} T_pM \cong U_i \times \mathbb{R}^n$, where (x^μ) are coordinates on U_i, and we parametrize a tangent vector as

$$\underbrace{v}_{\text{}} = v^\mu \frac{\partial}{\partial x^\mu} . \qquad (7.17)$$

coordinates on $T_pM \cong \mathbb{R}^n \quad \forall p \in U_i$

• A (smooth) tangent vector field is then (in coordinates)

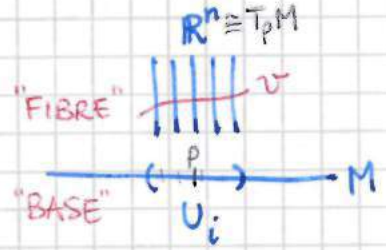
$$v = v^\mu(x)\frac{\partial}{\partial x^\mu}\,, \qquad (7.18)$$

with components $v^\mu(x)$ varying smoothly as p varies over M.
or rather over a patch U if we work in coords.

*** Ex 22:**
Check that (7.18) maps (locally) smooth functions to smooth functions, is linear, and obeys the Leibniz rule.

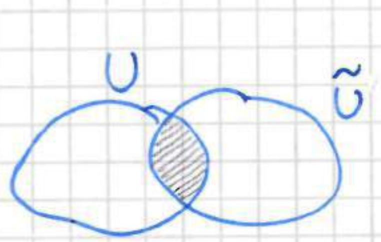
• **VOCABULARY:**
We say that a (smooth) tangent vector field v is a (smooth) "SECTION" of TM.
The reason for the term is as follows: (Write $v \in \Gamma(TM)$.)

- Locally, TM is a product space $U_i \times \mathbb{R}^n$.



- The vector field v draws a graph
  $(x^\mu, v^\nu(x))$ in $\varphi(U_i)\times\mathbb{R}^n$,
  which cuts the tangent bundle TM along the direction of M.
  Hence the name "section".

• This was a local description in a coordinate patch.
When we change patch from U to $\tilde{U}$ (on the overlap $U \cap \tilde{U}$), the coordinates on M change as



$$x^\mu \longmapsto \tilde{x}^\mu = \tilde{x}^\mu(x) \qquad (7.19)$$

$$x = \varphi(p) \longmapsto \tilde{x} = \tilde{\varphi}(p) = (\tilde{\varphi}\circ\varphi^{-1})(x)$$

In addition, we require the tangent space coordinates v^μ to change as

$$v^\mu \longmapsto \tilde{v}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\, v^\nu\,, \qquad (7.20)$$

so that

$$v = v^\mu \frac{\partial}{\partial x^\mu} = \tilde{v}^\mu \frac{\partial}{\partial \tilde{x}^\mu} \tag{7.21}$$

is independent of the choice of coordinates.

Proof:

$$\frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\nu} \Rightarrow v^\mu \frac{\partial}{\partial x^\mu} = v^\mu \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \frac{\partial}{\partial \tilde{x}^\nu} = \tilde{v}^\nu \frac{\partial}{\partial \tilde{x}^\nu}. \tag{7.22}$$

- The dual vector space to T_pM, i.e. the vector space of linear functionals on T_pM, is called the COTANGENT SPACE of M at p, T_p^*M:

$$T_p^*M \ni \omega : \begin{array}{l} T_p(M) \longrightarrow \mathbb{R} \\ \quad v \longmapsto \omega(v) \end{array} \tag{7.23}$$

s.t. $\forall a_1, a_2 \in \mathbb{R}$ $\forall v_1, v_2 \in T_pM$,

$$\omega(a_1 v_1 + a_2 v_2) = a_1 \omega(v_1) + a_2 \omega(v_2). \tag{7.24}$$

A dual basis to $\left\{\frac{\partial}{\partial x^\mu}\right\}$ for T_pM is $\{dx^\mu\}$ for T_p^*M, where we require
derivatives differentials

$$dx^\mu \left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu . \tag{7.25}$$

So we can write any cotangent vector $\omega \in T_p^*M$ as

$$T_p^*M \ni \omega = \omega_\mu dx^\mu . \tag{7.26}$$

Under a change of coord's on M (7.19) we require that

$$\omega_\mu \longmapsto \tilde{\omega}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \omega_\nu , \tag{7.27}$$

so that

$$\omega = \omega_\mu \, dx^\mu = \tilde{\omega}_\mu \, d\tilde{x}^\mu \tag{7.28}$$

is independent of the choice of coord's.

* Ex 23 :

1) Use the definition $df(x) = \frac{\partial f}{\partial x^\mu} dx^\mu$ of the differential of a function to show that under a coord. change (7.19)

$$dx^\mu \longmapsto d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \, dx^\nu \tag{7.29}$$

and therefore

$$\omega = \omega_\mu \, dx^\mu \longmapsto \tilde{\omega} = \tilde{\omega}_\mu \, d\tilde{x}^\mu = \omega_\nu \, dx^\nu = \omega \ . \tag{7.30}$$

2) Let $v = v^\mu \frac{\partial}{\partial x^\mu} \in T_p M$ and $\omega = \omega_\mu dx^\mu \in T_p^* M$. Show that

$$\omega(v) = \omega_\mu v^\mu \tag{7.31}$$

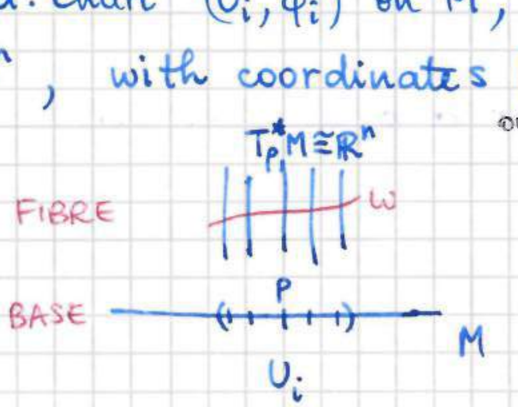
and that it is independent of the choice of coord's:

$$\omega_\mu v^\mu = \tilde{\omega}_\mu \tilde{v}^\mu \tag{7.32}$$

- We can construct the COTANGENT BUNDLE

$$T^* M = \bigcup_{p \in M} T_p^* M \tag{7.33}$$

as a collection of cotangent spaces over every point on M.
We require that for each coord. chart (U_i, φ_i) on M, $T^* U_i = \bigcup_{p \in U_i} T_p^* M \cong U_i \times \mathbb{R}^n$, with coordinates $(\underbrace{x^\mu}_{\text{on } U_i}, \underbrace{\omega_\mu}_{\text{on } \mathbb{R}^n})$.
Under a change of coord's,
we have (7.19) and (7.27), so
$\omega = \omega_\mu dx^\mu$ is coordinate independent.



A (smooth) cotangent vector field (covector) is (in coord's) → or (differential) 1-form

$$\omega = \omega_\mu(x)\, dx^\mu \qquad (7.34)$$

where $\omega_\mu(x)$ are smooth fn's. It's a (smooth) "section" of the cotangent bundle T^*M. (One writes $\omega \in \Gamma(T^*M)$.)

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• Remarks: (QUICK)

1) In Lagrangian mechanics

$$\underset{\substack{\text{generalised}\\ \text{coordinates}\\ (\text{base } M)}}{(q^i} \;,\; \underset{\substack{\text{generalised}\\ \text{velocities}\\ (\text{fibre} \cong T_pM)}}{v^j)}$$

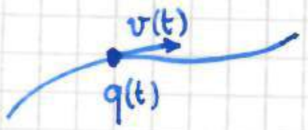
$v = v^j \frac{\partial}{\partial q^j}$ tangent vector

are coord's on the tangent bundle TM of the configuration space M

Under time evolution,

$(q^i(t))$ curve in M

$v^j(t) = \dot{q}^j(t)$ components of tangent vector to the curve



$v = v^j \frac{\partial}{\partial q^j}$ tangent vector

2) In Hamiltonian mechanics

$$\underset{\substack{\text{generalised}\\ \text{coordinates}\\ (\text{base } M)}}{(q^i} \;,\; \underset{\substack{\text{generalised}\\ \text{momenta}\\ (\text{fibre} \cong T^*_pM)}}{p_j)}$$

are coord's on the cotangent bundle T^*M of the config. space, where $p_j = \frac{\partial L}{\partial \dot{q}^j} = \frac{\partial L}{\partial v^j}$. Now $\theta = p_j\, dq^j$ is a cotangent vector, and one can write

$$H = L - \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} = L - \theta(v) .$$

DIGRESSION: (skip in view of time)

- A METRIC g on M is a symmetric bilinear form on TM which varies smoothly over M. At a point

non-degenerate: $g(\cdot, v) = 0 \Leftrightarrow v = 0$

$$g_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(u, v) \longmapsto g(u,v) = g(u,v)$$

In local coordinates

$$g = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu$$

A pair (M, g) is called a (PSEUDO) RIEMANNIAN MANIFOLD.

(PSEUDO): If g has negative eigenvalues

- A SYMPLECTIC FORM ω on M is a non-degenerate antisymmetric bilinear form on TM which varies smoothly over M. At a point

$$\omega_p : T_pM \times T_pM \longrightarrow \mathbb{R}$$
$$(u, v) \longmapsto \omega(u,v) = -\omega(v,u) .$$

In local coordinates

$$\omega = \omega_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu .$$

A pair (M, ω) is called a SYMPLECTIC MANIFOLD. ($\dim M = 2m$)

- EXAMPLES:
  - $(\mathbb{R}^n, g = \delta_{\mu\nu})$, $(\mathbb{R}^{1,3}, g = \eta_{\mu\nu})$ are (pseudo)Riemannian manifolds
  - Phase space in classical mechanics is a symplectic manifold, with Poisson bracket

$$\{f, g\} = \omega^{\mu\nu}\, \partial_\mu f \cdot \partial_\nu g .$$

In canonical coordinates (q^μ, p_μ):

$$[\omega_{\mu\nu}] = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

* Ex 24:

1) Show that requiring $g(u,v)$, $\omega(u,v)$ to be independent of the choice of coordinates requires that under a change of coordinates (7.19)

$$g_{\mu\nu} \longmapsto \tilde{g}_{\mu\nu} = \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}$$

$$\omega_{\mu\nu} \longmapsto \tilde{\omega}_{\mu\nu} = \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} \omega_{\rho\sigma}$$

2) Show that, given a tangent vector (field) v,

$$g(v,\cdot) : w \longmapsto g(v,w)$$

$$\omega(v,\cdot) : w \longmapsto \omega(v,w)$$

define a cotangent vector field, and that in coordinates

$$g(v,\cdot) = g_{\mu\nu} v^\mu dx^\nu$$

$$\omega(v,\cdot) = \omega_{\mu\nu} v^\mu dx^\nu$$

- For us, in $(M=\mathbb{R}^{1,3}, g=\eta)$, we denote $\eta_{\mu\nu} v^\mu := v_\nu$ and use the Minkowski metric η to lower indices (geometrically: to define an isomorphism between TM and T^*M).

* Ex 25:

1) Show that for $\mathbb{R}^2$ with Euclidean metric, changing from Euclidean coordinates to polar coordinates

$$x = (x^1, x^2) \longmapsto \tilde{x} = (r, \varphi) = \left(\sqrt{(x^1)^2+(x^2)^2}, \arctan\frac{x^2}{x^1}\right)$$

changes the metric as follows:

$$g = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 = dr \otimes dr + r^2 d\varphi \otimes d\varphi .$$

2) Repeat the exercise for $\mathbb{R}^3$.

7.2 – Fibre bundles

- We can generalize the previous construction by replacing T_pM or T_p^*M by a more general fibre.
- Simplest generalization: <u>vector bundle E</u>

$$\text{base} \quad M = \bigcup_{i\in I} U_i \qquad (\dim M = n)$$
$$\text{fibre} \quad F = V \ \underline{\text{vector space}} \qquad (\dim V = m)$$

Locally, E looks like $U_i \times V$, with coords (x, v).

<u>Def</u> – A diff'ble manifold E is a (smooth) <u>VECTOR BUNDLE</u> if

1) $\exists$ <u>projection map</u>

$$\pi : E \longrightarrow M \qquad (7.35)$$

s.t. $\forall p \in M, \quad \pi^{-1}(p) \cong V$. (fixed vector space) (7.36)

(This is a way of saying that M is part of E, and that for each point of M we have a vector space V.)

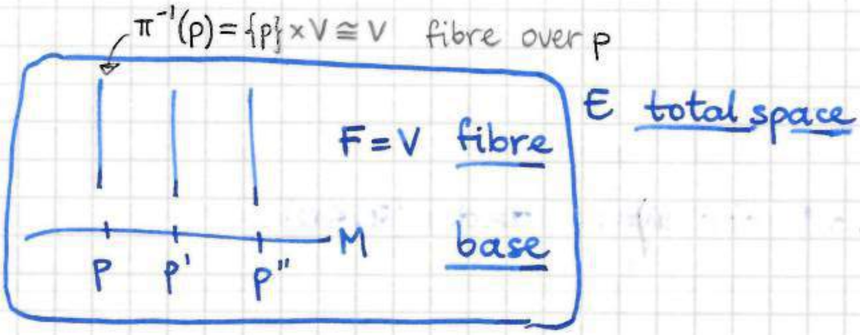
2) $\exists$ atlases of E and M s.t. $\forall$ charts U of M
$\exists$ a smooth map

$$\varphi : \pi^{-1}(U) \longrightarrow U \times V . \qquad (7.37)$$

"LOCAL TRIVIALISATION" of E over M

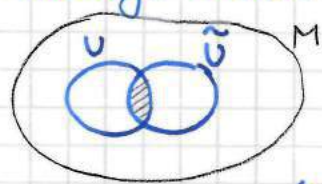
This means that we can use local coord's (x, v) for E,
where $\quad x$: coords over M
$\quad v \in V$ element of V associated to point x.

In pictures



(To be more precise, (E, M, π, V) is the vector bundle.)

- What happens to the fibre coord's when we change coord's in the base?



$$
\begin{aligned}
x^\mu &\longmapsto \tilde{x}^\mu = \tilde{x}^\mu(x) \\
v &\longmapsto \tilde{v} = t(x)\, v
\end{aligned}
\tag{7.38}
$$

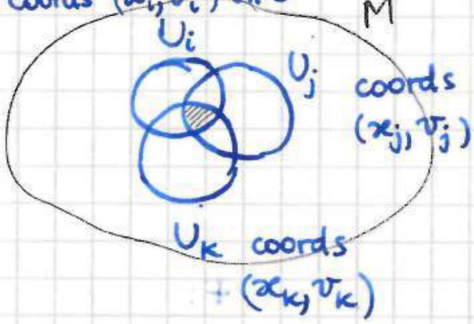
where the transition function for the fibre

$$t(x) \in GL(V) \equiv GL(m, \mathbb{R}) \tag{7.39}$$

(if V is a vector space over $\mathbb{R}$)

is an x-dependent invertible linear transformation.

- There is a consistency condition associated to triple overlaps $U_i \cap U_j \cap U_k$, which ensures that E is unique:



$$t_{k\leftarrow i}(x_i) = t_{k\leftarrow j}(x_j(x_i))\, t_{j\leftarrow i}(x_i) \qquad \text{COCYCLE CONDITION} \tag{7.40}$$

This ensures the compatibility of

$$
\begin{aligned}
k \leftarrow i &: \quad v_k = t_{k\leftarrow i}(x_i)\, v_i \\
k \leftarrow j \leftarrow i &: \quad v_k = t_{k\leftarrow j}(x_j)\, v_j = t_{k\leftarrow j}(x_j)\, t_{j\leftarrow i}(x_i)\, v_i .
\end{aligned}
\tag{7.41}
$$

* **Ex 26**:

Show that the transition functions for TM and T^*M obey (7.39) for $m=n$ and (7.40).

[Hint: chain rule.]

• **REMARKS**:

1) Unlike for TM and T^*M, the transition functions for the fibre of a general vector bundle are independent of the transition fns on the base.

2) We could take $\tilde{x}^\mu = x^\mu$ (no change of coords in the base) but still change coords in the fibre. (7.39), (7.40) must still hold.

VOCABULARY: "LINE BUNDLE" is a (usually cplx) vector bundle with fibre of dimension 1) E 29

• We can generalize further if we allow the fibre F to be a more general object rather than a vector space. For us: F will be a diff'ble manifold itself.

Def - A diff'ble manifold E is a FIBRE BUNDLE if

1) $\exists$ projection map

$$\pi : E \to M \qquad M: \text{BASE} \tag{7.42}$$

s.t.

$$\forall p \in M \quad \pi^{-1}(p) \cong F. \qquad F: \text{FIBRE} \tag{7.43}$$

2) $\exists$ atlases of E and M s.t. $\forall$ charts U of M $\exists$ a smooth map

$$\varphi : \pi^{-1}(U) \to U \times F \qquad \text{LOCAL TRIVIALISATION of } E \text{ over } M \tag{7.44}$$

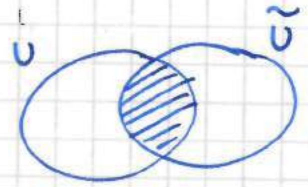
Local coords : (x, y)
base M fibre F

When we change coord's in the base M, the fibre coord's must change appropriately, and the transition functions for the fibre must obey a cocycle condition. The transition fns for the fibre are elements of a group, called the STRUCTURE GROUP of the bundle E.

*** EXAMPLE : PRINCIPAL G-BUNDLE P**

Let $F = G$ Lie group, e.g. $U(1), SO(2), SO(3)$.

Coordinates $(x, \overset{\in G}{h})$ on $U \times G$
$(\tilde{x}, \tilde{h})$ on $\tilde{U} \times G$



Then we require the transition function $t(x)$ to be a group element $g(x)$ itself : (so the fibre is $F = G$, and the structure group is also G)

$$(x^\mu, h) \longmapsto (\tilde{x}^\mu(x), \tilde{h} = \underset{\substack{\cap \\ G}}{g(x)}\, h) \qquad (7.45)$$

· REMARKS:

1) This is called a "principal" bundle because of its importance. It controls the structure of (infinitely) many vector bundles. Indeed, for each representation $\underline{r}$ of G, we have a vector space $V^{(r)}$ of dim. r and an action of G on $V^{(r)}$ by a representation matrix $r(g)$. (technically : $P \times_G V^{(r)} = \frac{P \times V^{(r)}}{G}$)
We can then define an ASSOCIATED VECTOR BUNDLE E with

$$\text{fibre } F = V^{(r)} \ni v$$
$$\text{transition functions } \; t(x) = r(g(x)) \qquad (7.46)$$

so

$$(x, v) \longmapsto (\tilde{x}(x), \tilde{v} = r(g(x))v) \qquad (7.47)$$

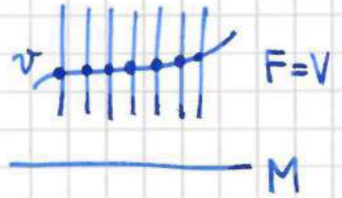
2)

Maths	Physics
Principal G-bundle	Gauge symmetry G
(Section of) Associated vector bundle	Charged field

* Ex 27: U(1) bundles over S^2 (Monopole bundle / Hopf bundle)

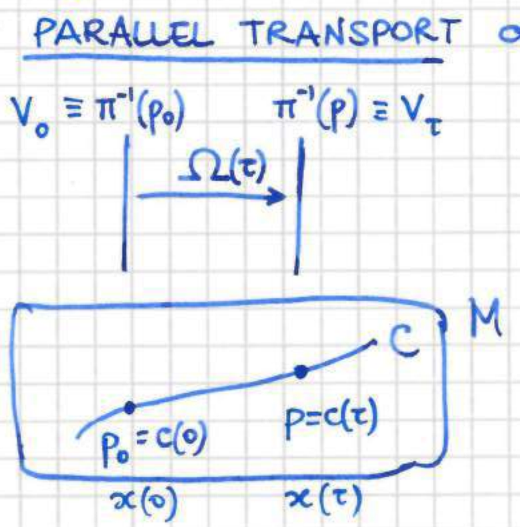
7.3 - Connection, holonomy and curvature

• Let $v(x)$ be a section of a vector bundle over M, in local coordinates.



Can we define partial derivatives of v, or directional derivatives along a curve C in M (in coords, $x^\mu = x^\mu(\tau)$)?

We can't subtract vectors defined at infinitesimally close points, because they belong to different vector spaces.
We need a way of comparing vectors defined at different points along the curve C. We can do this by defining the notion of PARALLEL TRANSPORT along C, through a linear map



$$\Omega(\tau): \quad V_0 \longrightarrow V_\tau$$
$$v_0 \longmapsto \Omega(\tau) v_0 \tag{7.48}$$

which is invertible and obeys $\Omega(0) = \mathbb{1}$. More generally,

$$\Omega(\tau')\,\Omega^{-1}(\tau) : \quad V_\tau \longrightarrow V_{\tau'} \tag{7.49}$$

can be used to compare vectors defined at $x(\tau)$ to vectors defined at $x(\tau')$. By comparing infinitesimally close points $x^\mu = x^\mu(\tau)$ and $x^\mu(\tau + \varepsilon d\tau)$, we can define the COVARIANT DERIVATIVE $\nabla_\mu v$ by

ε is just a bookkeping device

$$\nabla v = \nabla_\mu v \, dx^\mu := \lim_{\varepsilon \to 0} \frac{v(x(\tau+\varepsilon d\tau)) - \Omega(\tau+\varepsilon d\tau)\,\Omega^{-1}(\tau)\, v(x(\tau))}{\varepsilon}, \tag{7.50}$$

where $dx^\mu = \dot{x}^\mu(\tau) d\tau$ in the parametrization of the curve.

• REMARK :

(7.50) only depends on the local (infinitesimal) form of Ω near τ. Letting

$$\Omega(\tau + \varepsilon d\tau) = \Omega(\tau) - \varepsilon\, \mathcal{A}(x(\tau))\,\Omega(\tau) + O(\varepsilon^2), \tag{7.51}$$

(7.50) becomes (where $x(\tau) \equiv x$)

$$\nabla v(x) = \mathbf{d} v(x) + \mathcal{A}(x)\, v(x), \tag{7.52}$$

where the CONNECTION $\mathcal{A}(x)$ is a matrix-valued(*) cotangent vector field (or differential 1-form):

$$\mathcal{A}(x) = \underbrace{\mathcal{A}_\mu(x)}_{\text{matrix}} dx^\mu \tag{7.53}$$

(*) Formally, it takes value in the Lie algebra of the structure group of the vector bundle.

In components,

$$\nabla_\mu v^\alpha(x) = \partial_\mu v^\alpha(x) + \mathcal{A}_\mu(x)^\alpha{}_\beta\, v^\beta(x) . \qquad (7.54)$$

The connection $\mathcal{A}$ encodes the infinitesimal version of parallel transport.

- Now consider a change of coordinates in the fibre only:

$$\underset{M}{(x,} \underset{V}{v)} \longmapsto (x, \tilde{v} = t(x)\, v) . \qquad (7.55)$$

We want ∇v to transform like v, so

$$\nabla_\mu v(x) \longmapsto t(x) \nabla_\mu v(x) , \qquad (7.56)$$

or in terms of differential operators

$$\nabla_\mu \longmapsto t \nabla_\mu t^{-1} . \qquad (7.57)$$

This requires the connection to transform as

$$\mathcal{A}_\mu \longmapsto \tilde{\mathcal{A}}_\mu = t\, \partial_\mu t^{-1} + t\, \mathcal{A}_\mu t^{-1} . \qquad (7.58)$$

- REMARKS:

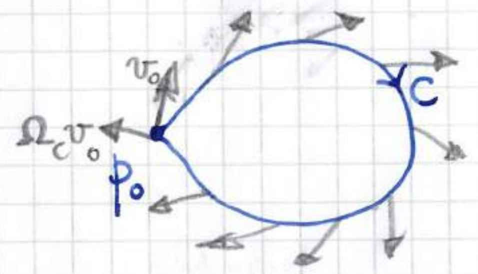
1) This construction works for any vector bundle E (any fibre bundle in fact).

2) When E is an associated vector bundle to a principal G-bundle, $\mathcal{A}_\mu = -i A_\mu^{(r)}$ with $A_\mu^{(r)}$ the gauge field in the appropriate representation. (For a principal G-bundle, $A_\mu \equiv i\mathcal{A}_\mu$ is the Lie algebra-valued gauge field which transforms as $A_\mu \mapsto g \partial_\mu g^{-1} + i g A_\mu g^{-1}$.)

3) When E is the tangent bundle, $\mathcal{A}_\mu$ is the affine connection that appears in general relativity.

Next, holonomy and curvature.

• Next, we go back to the finite version of parallel transport. Consider a closed path (or loop) C, starting and ending at the same point p_0 (base point). We can parallel transport a vector



$v_0 \in \pi^{-1}(p_0)$ along the loop C to get a new vector $\Omega_C v_0 \in \pi^{-1}(p_0)$,

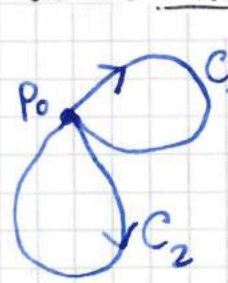
(Here $\Omega_C \equiv \Omega(\tau_f)$, where $x^\mu(\tau_i) = x^\mu(\tau_f) = x^\mu_0$. $\Omega(\tau_i) = \mathbb{1}$)

which is "rotated" by

$$\Omega_C \in GL(V) . \qquad \frac{\text{HOLONOMY}}{\text{ALONG } C} \qquad (7.59)$$

• Holonomies along closed paths starting and ending at the same base point p_0 form a group, called the HOLONOMY GROUP, which is a subgroup of $GL(V)$.
(This is a consequence of the fact that closed paths form a group, more about this later.)

– If we concatenate two loops C_1 and C_2 to get the loop $\underset{\text{second}}{C_2} \circ \underset{\text{first}}{C_1}$,



we get

$$\Omega_{C_2 \circ C_1} = \Omega_{C_2} \Omega_{C_1} \qquad (7.60)$$

which is the composition law ("product") in the group.

– The homotopy along the trivial loop, which doesn't move from the base point p_0, is the identity element in the group.

– Given a loop C, we can define the loop $-C$ with opposite orientation. Then

$$\Omega_{-C} = \Omega_C^{-1} \qquad (7.61)$$

is the inverse element to Ω_c in the holonomy group.

• REMARK:

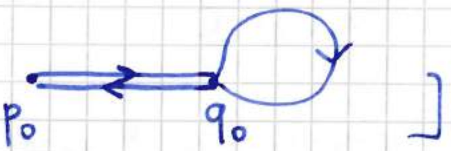
The holonomy group is (generically) non-abelian:

$$\Omega_{C_1}\,\Omega_{C_2} \neq \Omega_{C_2}\,\Omega_{C_1}\,. \qquad (7.62)$$

* Ex 28:

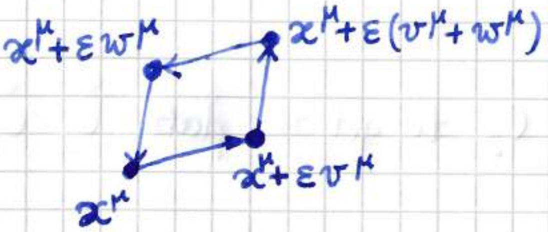
Let M be connected, that is, any two points $p_0, q_0 \in M$ can be connected by a path in M. Show that the holonomy groups based at p_0 and q_0 are isomorphic.

[Hint:



]

• The CURVATURE $\mathcal{F}_{\mu\nu}$ is the holonomy along an infinitesimal loop. More precisely, consider the loop dC:



Then

$$\Omega_{dC} = \mathbb{1} + \varepsilon^2 \mathcal{F}_{\mu\nu}(x)\, v^\mu w^\nu + \mathcal{O}(\varepsilon^3) \qquad (7.63)$$

where

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]. \qquad (7.64)$$

Proof: * Ex 29.

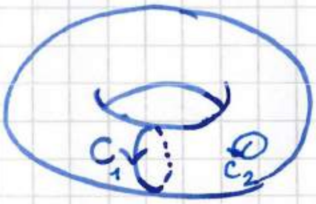
Under a change of coordinates in the fibre (7.55), the curvature transforms as follows:

$$\mathcal{F}_{\mu\nu} \longmapsto \widetilde{\mathcal{F}}_{\mu\nu} = t\, \mathcal{F}_{\mu\nu}\, t^{-1} . \qquad (7.65)$$

• <u>REMARKS</u>:

1) For a principal G-bundle, $\mathcal{F}_{\mu\nu} = -iF_{\mu\nu}$, where $F_{\mu\nu}$ is the field strength of A_μ. (For an associated vector bundle, $\mathcal{F}_{\mu\nu} = -i\,F^{(\Omega)}_{\mu\nu}$.)

2) Even if the curvature / field strength is 0, the holonomy can be non-trivial if the loop C is not contractible to a point. E.g. on a torus T^2 (the surface of a donut)



the holonomy along C_1 (not contractible) need not be trivial, whereas the holonomy along C_2 (contractible) is trivial ($\Omega_{C_2} = \mathbb{1}$) if $\mathcal{F}_{\mu\nu} = 0$. [*Ex 30]

• <u>VOCABULARY</u>: if $\mathcal{F}_{\mu\nu} = 0$, we say that the connection $\mathcal{A}_\mu$ is <u>flat</u> (or that the bundle is flat).

The holonomy of a flat connection is called <u>monodromy</u>.

- **SUMMARY:**

Maths	Physics
Principal G-bundle	Gauge symmetry G
Connection A_μ of Pr. G-bundle	Gauge field A_μ
Curvature $F_{\mu\nu}$ " "	Field strength $F_{\mu\nu}$
(Section of) Associated vector bundle	Charged field
Covariant derivative ∇_μ	Covariant derivative D_μ
Parallel transport	Wilson line
$\mathrm{tr}(\Omega_c)$ holonomy	Wilson loop