8. TOPOLOGICAL SOLITONS & INSTANTONS

This chapter: non-trivial solutions of field equations whose existence and stability is ensured by topology.

8.1 – Global vortices

[E. Weinberg 3.1; Manton-Sutcliffe 7.1, 7.2]

- Vortices are topological solitons (non-trivial finite energy localised sol'ns protected by topology) which look like particles in 2 SPACE + 1 TIME dimensions.

(They look like strings in 3+1, membranes in 4+1, etc)

Coordinates $(\underset{t}{\underset{\shortparallel}{x^0}}, x^1, x^2)$. Let

$$z = x^1 + i x^2 = r e^{i\theta} \tag{8.1}$$

be a complex coord. for $\mathbb{C} \cong \mathbb{R}^2$ (space).

- Consider a cplx scalar field ϕ with a U(1)-symmetric scalar potential

$$V(\phi,\bar\phi) = U(|\phi|^2) = \frac{\lambda}{2}\left(|\phi|^2 - v^2\right)^2 . \qquad \overset{\text{constants}}{\left(\begin{matrix} v>0 \\ \lambda>0 \end{matrix}\right)} \tag{8.2}$$

The ENERGY of a static field config'n ($\dot\phi = 0$) is

$$E = \int d^2x \left[\underbrace{|\nabla\phi|^2}_{\substack{\equiv \\ (\partial_j\bar\phi)(\partial_j\phi) \\ \text{or} \\ \sum_{j=1}^{2} |\partial_j\phi|^2}} + \underbrace{U(|\phi|^2)}_{\substack{= \\ \frac{\lambda}{2}(|\phi|^2 - v^2)^2 \\ \geq \\ 0}} \right] \tag{8.3}$$

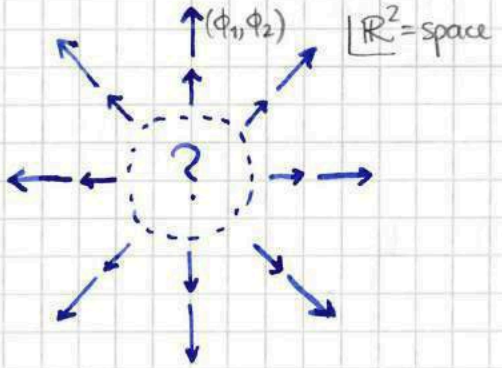
Static
• Configurations of minimum energy $\overset{E=0}{}$ (<u>vacua</u>)

$$\phi = |\phi| e^{i \arg(\phi)} = v e^{i\alpha} \qquad (8.4)$$

where $\alpha \sim \alpha + 2\pi$ is a constant. They parametrize the vacuum manifold (or "moduli space of vacua")

$$V = S^1 \qquad \text{(circle)} \qquad (8.5)$$

• We are interested in less trivial static configurations of finite energy, which look like this: (drawing $\vec{\Phi} = (\mathrm{Re}\,\phi, \mathrm{Im}\,\phi)$ in $\mathbb{R}^2$ = space)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \overset{|||}{\Phi_1} \qquad \overset{|||}{\Phi_2}$



REMARKS:

1) Such a field config'n cannot be deformed continuously to the vacuum, if we keep the boundary conditions (BC) at spatial infinity fixed.

2) Keeping the BC fixed, the above static field configuration can be deformed continuously until the energy (8.3) is minimized. The resulting configuration solves the (E.-L.) EoM (* Ex 31) and is called a GLOBAL VORTEX.
$\updownarrow$
because U(1) GLOBAL symmetry.

• There is some interesting underlying topology.
Given a smooth config'n of ϕ, we can define the VORTICITY /VORTEX NUMBER/WINDING NUMBER

$$N[C] = \frac{1}{2\pi}\oint_C \nabla \arg(\phi)\cdot d\vec{l} = \frac{1}{2\pi}\oint_C \partial_i \arg(\phi)\, dx^i \qquad (8.6)$$

associated to a loop C.

1) If $\phi \neq 0$ along C, $\arg(\phi)$ is well defined and (8.6) counts the number of full rotations that $\arg(\phi)$ makes during one clockwise circuit along C, which is an <u>integer</u>.

2) If we deform C continuously, $N[C]$ also changes continuously. But it's an integer, so it can only be continuous by being a <u>constant</u>. (One says that $N[C]$ is "topologically conserved")

3) There is an <u>exception</u>: if C passes through a point P where $\phi = 0$, then $\arg(\phi)$ is ill-defined, and so is (8.6). Then

$$N[C] - N[C'] = N[C_P] \qquad (8.7)$$

and the jump in the vortex number, $N[C_P]$, is the order of vanishing of ϕ at P.

(If $N[C_P] < 0$, then $-N[C_P]$ is the order of a pole).

* <u>Ex 32</u>: Let $\phi \approx \overset{\text{constant}}{c}\cdot(z-z_0)^n$ near $z = z_0$. Show that $N[C_{z_0}] = n$ if C_{z_0} is an infinitesimal loop encircling z_0.

We say that there are:

$N[C_P]$ <u>vortices</u> sitting at P if $N[C_P] > 0$

$-N[C_P]$ <u>antivortices</u> " " " $N[C_P] < 0$.

4) So

$$N[C] = \left(\underset{\text{enclosed by } C}{\#\text{ vortices}}\right) - \left(\underset{\text{enclosed by } C}{\#\text{ antivortices}}\right) \tag{8.8}$$

and, letting $C = S^1_\infty$, the circle at spatial infinity,

$$N \equiv N[S^1_\infty] = \begin{pmatrix}\text{total}\\ \#\text{ vortices}\end{pmatrix} - \begin{pmatrix}\text{total}\\ \#\text{ antivortices}\end{pmatrix}. \tag{8.9}$$

5) Topology is the branch of maths that studies shapes up to continuous deformations. Topologically, $\underset{\text{loop}}{C} \simeq \underset{\text{circle}}{S^1}$ and

$$\arg(\phi)\big|_C : \underset{\substack{\text{spatial}}}{C} \longrightarrow \underset{\text{field space / "target space"}}{S^1}, \qquad C \cong S^1$$

counts how many times the spatial loop C winds around the circle parametrized by $\arg(\phi)$ in field space / target space. Maps from S^1 to S^1 are labelled by an integer, the winding number $N[C]$.

- Let's now look for a static solution of the field equations

$$\nabla^2\phi - \lambda\left(|\phi|^2 - v^2\right)\phi = 0 \tag{8.10}$$

with $N=1$. Solving (8.10) in general is hard, so let's make the ansatz

$$\phi(\underline{x}) = f(r)\, e^{i\theta}. \tag{8.11}$$

[NOTE: (8.11) is invariant under a combined rotation in spatial $\mathbb{R}^2$ and a phase rotation of ϕ.]

If we require ϕ to be invariant under a reflection about the x^1-axis combined with cplx. conjugation of ϕ, then

$$\phi(r,\theta) = \phi^*(r,-\theta) \tag{8.12}$$

which implies

$$f(r) \in \mathbb{R} . \tag{8.13}$$

Subbing (8.11)-(8.13) in (8.10), we get the <u>ODE</u>

$$f''(r) + \frac{1}{r} f'(r) - \frac{1}{r^2} f + \lambda(v^2 - f^2) f = 0 . \tag{8.14}$$

<u>Proof</u> : * Ex 33 .

We impose the <u>BOUNDARY CONDITIONS</u>

$$f(0) = 0 , \qquad f(\infty) = v \tag{8.15}$$

so that ϕ is non-singular at the origin and $U(|\phi|^2) \longrightarrow 0$ at spatial infinity.

The eqn (8.14) with BC (8.15) can be solved numerically, but we run into a problem: the static energy (8.3) is <u>infinite</u> because $\nabla\phi \not\to 0$ at spatial infinity, due to the θ-dependence in (8.11).

$\boxed{E \ldots}$

<u>* Ex 34</u> :

1) Let $\phi(\underline{x}) = \rho(\underline{x})\, e^{i\alpha(\underline{x})}$. Show that

$$E = \int d^2x \left[(\nabla\rho)^2 + \rho^2 (\nabla\alpha)^2 + \frac{\lambda}{2}(\rho^2 - v^2)^2 \right] .$$

2) Let $\phi = f(r) e^{i\theta}$ as in (8.11). Show that

$$\rho^2 (\nabla\alpha)^2 = \frac{f^2}{r^2} \xrightarrow[r\to\infty]{} \frac{v^2}{r^2} ,$$

and that this causes a logarithmic divergence of E.
(That is, let $E_R = \int_{r\le R} d^2x \ldots$ and show that $E_R \sim \log R$ as $R \to \infty$.)

8.2 – Derrick's theorem

[E. Weinberg 3.2]

- The previous negative result is not an accident, but a particular case of a more general theorem due to Derrick (1964).

- Consider a <u>scalar field theory</u> in <u>D space dimensions</u>:

$$\mathcal{L} = -\frac{1}{2} G_{ab}(\phi)\, \partial_\mu \phi^a \partial^\mu \phi^b - V(\phi) \,, \tag{8.16}$$

where the matrix $G_{ab}(\phi)$ is positive definite $\forall \phi$ and $V(\phi) = 0$ at its global minima.

Then any <u>finite energy static solution</u> of the field equations is a <u>stationary</u> point of the <u>static (or potential) energy</u> functional

$$E[\phi] = E_K[\phi] + E_V[\phi] \,,$$

$$E_K[\phi] = \int d^D x \, \frac{1}{2} G_{ab}(\phi)\, \partial_i \phi^a \partial_i \phi^b \quad \geq 0 \quad \text{because } G_{ab}(\phi) \text{ +ve definite} \tag{8.17}$$

$$E_V[\phi] = \int d^D x \; V(\phi) \,. \quad \geq 0 \quad \text{because } V(\phi) \geq 0$$

[Spatial indices i are summed over in $E_K[\phi]$ – no distinction between upper and lower indices. Internal indices a, b are also summed over.]

* <u>Ex 35.1</u>: prove the claim above (8.17).
[<u>Hint</u>: treat $E[\phi]$ as an action for static field config's, and derive its Euler-Lagrange eqns.]

- Given a static solution of the EoM $\phi(x) = \phi_1(x)$, consider

$$\phi(x) = \phi_\lambda(x) \equiv \phi_1(\lambda x) \,, \tag{8.18}$$

a <u>1-parameter family</u> of field config's labelled by <u>$\lambda > 0$</u>.

Then

$$E[\phi_\lambda] = E_K[\phi_\lambda] + E_V[\phi_\lambda] \\ = \lambda^{2-D} E_K[\phi_1] + \lambda^{-D} E_V[\phi_1] \tag{8.19}$$

Proof : * Ex 35.2 .

We know that $\lambda = 1$, which returns the static sol'n of the EoM, is a stationary point of $E[\phi_\lambda]$, hence

$$\begin{aligned} 0 &= \frac{d}{d\lambda} E[\phi_\lambda]\Big|_{\lambda=1} \\ &= (2-D)\lambda^{1-D} E_K[\phi_1] - D\lambda^{-1-D} E_V[\phi_1]\Big|_{\lambda=1} \\ &= (2-D)\underbrace{E_K[\phi_1]}_{\geq 0} - D\underbrace{E_V[\phi_1]}_{\geq 0} \end{aligned} \tag{8.20}$$

- **D = 1** : we get the identity

$$E_K[\phi_1] = E_V[\phi_1] \quad . \tag{8.21}$$

There can be non-trivial finite energy solutions of the field eqns.

- **D = 2** :

$$E_V[\phi_1] = 0 , \tag{8.22}$$

which implies that ϕ_1 must be a vacuum for all x.
In particular, This rules out a global vortex solution, since it's not a vacuum. (need $\phi = 0$ at the core)

- **D > 2** : the two terms in the last line of (8.20) have the same sign. The only solution to (8.20) is $E_K[\phi_1] = E_V[\phi_1] = 0$, which again means that ϕ_1 must be a vacuum everywhere.

CONCLUSION :

There are no non-trivial (=non-constant) finite energy static solutions of the scalar field theory (8.16) in $D > 1$ space dimensions.

WAY OUT : introduce gauge fields.

8.3 - Gauged vortices

[E. Weinberg 3.3
Manton, Sutcliffe 7.1, 7.3
Tong (GT) 2.5.2]

- Let's return to our model for vortices in 2+1 dimensions, but now with a U(1) gauge symmetry.
We obtain the Abelian Higgs model, with

$$\mathcal{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} - \overline{D_\mu\phi}\, D^\mu\phi \underbrace{- \frac{\lambda}{2}(|\phi|^2 - v^2)^2}_{= U(|\phi|^2)}, \qquad (8.23)$$

where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ D_\mu\phi &= \partial_\mu\phi - iA_\mu\phi . \end{aligned} \qquad (8.24)$$

- In the gauge $A_0 = 0$, the energy of static field configurations ($\dot{\phi}=0$, $\dot{A}_i=0$) is

$$E = \int d^2x \left[\sum_{j=1}^{2} |D_j\phi|^2 + \overbrace{\frac{\lambda}{2}(|\phi|^2 - v^2)^2}^{= U(|\phi|^2)} + \frac{1}{2g^2}B^2 \right] \qquad (8.25)$$

where

$$B = F_{12} = \partial_1 A_2 - \partial_2 A_1 . \qquad (8.26)$$

(In 3d: magnetic field transverse to the plane)

[See *Ex 6 in the problem sheet.]

- Remarks:

1) Vacua/Ground states are config's with $E=0$:

$$\begin{aligned} &(1) \quad D_j\phi = \partial_j\phi - iA_j\phi = 0 \\ &(2) \quad |\phi| = v \\ &(3) \quad B = 0 . \end{aligned} \qquad (8.27)$$

Eq. (2) sets $\phi = v e^{i\alpha(\underline{x})}$, and subbing in (1) we find

$\underline{x} = (x^1, x^2) = (x_1, x_2)$

$$A_j = -i\,\partial_j \log\phi = \partial_j \alpha\,, \qquad (\text{"pure gauge"}) \qquad (8.28)$$

which implies

$$B = F_{12} = 0\,. \qquad (8.29)$$

[...]

2) A necessary condition for $E < \infty$ is that the fields (ϕ, A) approach a vacuum at spatial infinity:

$$\lim_{|\underline{x}|\to\infty}\left(\phi(\underline{x}) - v\,e^{i\overset{\mathbb{R}}{\overset{\Downarrow}{\alpha_\infty(\underline{x})}}},\; A_j(\underline{x}) - \partial_j\,\alpha_\infty(\underline{x})\right) = 0\,. \qquad (8.30)$$

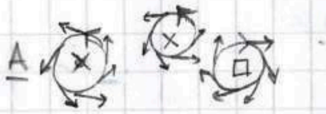
By the logic of (8.29), this implies that $B \to 0$ at infinity.

3) The total vortex number / winding number

$$N = N[S^1_\infty] := \frac{1}{2\pi}\oint_{S^1_\infty} \nabla\alpha_\infty \cdot \overset{d\ell}{d\underline{x}} = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, \frac{d\alpha_\infty(\theta)}{d\theta} \quad (\text{at } r\to\infty) \qquad (8.31)$$

is gauge invariant and proportional to the magnetic flux through the plane

$$N = \frac{1}{2\pi}\oint_{S^1_\infty} \underline{A}\cdot \overset{d\ell}{d\underline{x}} \underset{\text{Stokes}}{=} \frac{1}{2\pi}\int_{\mathbb{R}^2} B\, d^2x = \frac{\Phi(B)}{2\pi} \qquad (8.32)$$



X : Vortex
□ : antivortex

This means in particular that the magnetic flux through the spatial plane is quantized. (QM: There's a hidden $\hbar$ which I set to 1.)

4) In (8.31), the asymptotic profile $\alpha_\infty(\theta)$ of $\arg(\phi)$ defines a map

$$\alpha_\infty : S^1_\infty \longrightarrow S^1\,. \qquad (8.33)$$

Such maps fall into disjoint classes, labelled by the integer N, the winding number, which counts the number of times $\alpha_\infty(S^1_\infty)$ winds around S^1.

Mathematically, continuous maps from S^1 to S^1 are (partially) classified by the 1st homotopy group / "fundamental group" of S^1

$$\pi_1(S^1) = \underset{\substack{\cup \\ \mathbb{N}}}{\mathbb{Z}} . \tag{8.34}$$

- Can we find $\overset{\text{static}}{\text{solutions}}$ of the equations of motion with $N=1$?

In the gauge $A_0 = 0$, one can use the ansatz

$$\begin{aligned} \phi(\underline{x}) &= v\, e^{i\theta} f(vr) \\ A_j(\underline{x}) &= \varepsilon_{jk}\, \hat{x}^k \frac{a(vr)}{r} \end{aligned} \tag{8.35}$$

real

Remarks:

1) (8.35) is rotationally symmetric: the effect of a spatial rotation can be undone by a gauge transformation.

2) (8.35) is invariant under reflection about the x^1-axis combined with cplx conjugation (of ϕ.)

3) We require that

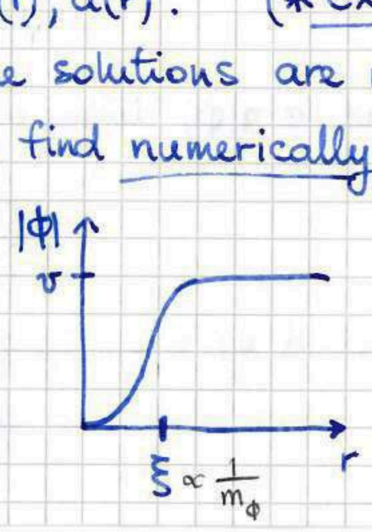
$$f, a \xrightarrow[r\to 0]{} 0 \quad , \quad f, a \xrightarrow[r\to\infty]{} 1 \tag{8.36}$$

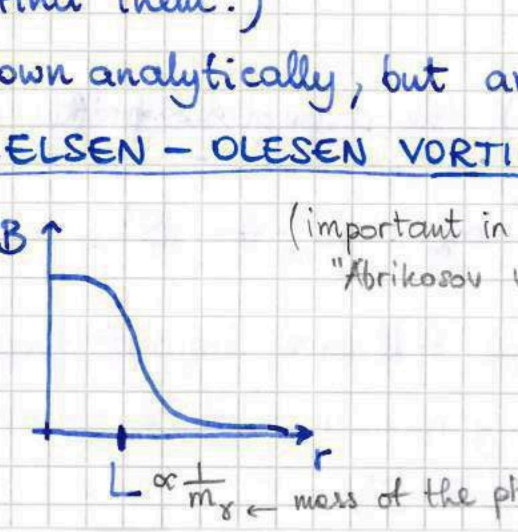
to ensure regularity at the origin and the BC at infinity.

4) Using (8.35), the EoM reduce to a system of 2 ODE's for $f(r)$, $a(r)$. (* Ex 36: find them.)

The solutions are not known analytically, but are easy to find numerically: NIELSEN – OLESEN VORTICES.

(important in superconductivity: "Abrikosov vortices")



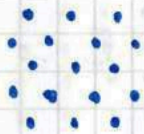


5) The physics of <u>multiple</u> vortices depends on the ratio between the length scales ξ and L associated to ϕ and B, due to competition between

- attractive force due to $\phi \approx 0$
- repulsive force due to $B \neq 0$

• <u>Type I ($\xi > L$)</u>: attraction wins $\longrightarrow$ separate vortices combine into one big vortex, inside which we are just in a normal vacuum.

• <u>Type II ($\xi < L$)</u>: repulsion wins $\longrightarrow$ lattice of separate vortices (<u>Abrikosov lattice</u>).



• <u>$\xi = L$</u> : <u>zero net force</u>. Coming next...

$\hookrightarrow$ <u>$\lambda = g^2$</u>, it turns out .

8.3.1 - Bogomol'nyi vortices

Let <u>$\lambda = g^2$</u>. We can massage the static energy (8.25), using a trick due to Bogomol'nyi, to find a lower bound for the energy for every fixed value of N:

$$E = \int d^2x \left[\sum_{j=1}^{2} |D_j\phi|^2 + \frac{g^2}{2}(|\phi|^2 - v^2)^2 + \frac{1}{2g^2} B^2 \right]$$

"complete the squares" (twice)

$$= \int d^2x \Big[|D_1\phi - i\varepsilon_1 D_2\phi|^2 + i\varepsilon_1 \left(\overline{D_1\phi}\, D_2\phi - \overline{D_2\phi}\, D_1\phi \right) + \frac{1}{2g^2}\left(B - \varepsilon_2 g^2 (|\phi|^2 - v^2) \right)^2 + \varepsilon_2 B (|\phi|^2 - v^2) \Big] \qquad (8.37)$$

where $\varepsilon_1^2 = \varepsilon_2^2 = 1$ (i.e., $\varepsilon_1, \varepsilon_2$ are signs).

Integrating by parts and using the BC to drop boundary/surface terms,

$$\int d^2x \left((D_1\bar{\phi})(D_2\phi) - (D_2\bar{\phi})(D_1\phi) \right) = -\int d^2x\, \bar{\phi}\,[D_1, D_2]\,\phi$$
$$= +\int d^2x\, i\,\bar{\phi}\, F_{12}\,\phi = i\int d^2x\, B|\phi|^2 . \tag{8.38}$$

Hence

$$E = \int d^2x \Big[\underbrace{|(D_1 - i\varepsilon_1 D_2)\phi|^2}_{\geq 0} + \frac{1}{2g^2}\underbrace{\left(B - \varepsilon_2 g^2(|\phi|^2 - v^2)\right)^2}_{\geq 0}$$
$$- \varepsilon_1 B|\phi|^2 + \varepsilon_2 B\left(|\phi|^2 - v^2\right) \Big] \tag{8.39}$$

Picking $\varepsilon_2 = \varepsilon_1 \equiv \varepsilon$, we find that

$$E \geq -\varepsilon v^2 \int d^2x\, B = -2\pi\varepsilon v^2 N . \tag{8.40}$$

Since this holds for $\varepsilon = \pm 1$, we find the <u>BOGOMOL'NYI BOUND</u>

static energy $\quad E \geq 2\pi v^2 |N| . \quad$ Vortex number $\quad$ (8.41)

• The static sol'ns that saturate the bound, i.e. $E = 2\pi v^2 |N|$, obey the 1st order <u>BOGOMOL'NYI EQNS</u>

$$(D_1 - i\varepsilon D_2)\phi = 0$$
$$B = \varepsilon g^2 \left(|\phi|^2 - v^2\right) \tag{8.42}$$

where $\varepsilon = -\mathrm{sign}(N)$. They are called <u>BOGOMOL'NYI VORTICES</u>.
(if $N>0$, otherwise antivortices)

The general solution can be found explicitly, and has $2N$ real parameters.

8.4 – The Dirac monopole

[Nakahara 1.9, (9.4.1, 10.5.2)]

- If we extend the Abelian Higgs model to 3+1 spacetime dimensions, static gauged vortices look like strings extending along x^3: the magnetic field is localised in the x^1-x^2 plane, near the core of the vortices.

$\boxed{Q}$ Can we have a magnetic field localised near /at a point in $\underset{\text{space}}{\mathbb{R}^3}$? (MAGNETIC MONOPOLE)
We can already ask the question in pure E-M.

$\boxed{A1}$ NO!

1) $\nexists$ magnetic charge $\overset{\text{current}}{\text{density}}$ in Maxwell's eqns

$$\begin{aligned} \partial_\mu F^{\mu\nu} &\propto j^\nu &&\leftarrow \text{electric current} \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 &&\leftarrow \text{no magnetic current} \end{aligned} \tag{8.43}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ $\quad (\varepsilon^{0123}=1)$.

2) In the gauge theory formulation of E-M, for static fields

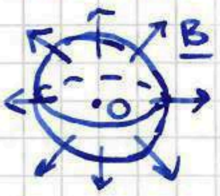
$$\underline{B} = \nabla \times \underline{A} \Rightarrow \nabla\cdot\underline{B} = \overset{\rho_m = 0}{0}. \tag{8.44}$$

- This seems to suggest that pointlike magnetic charges don't exist, but Dirac (1931) found a loophole in this reasoning. This was understood later in geometric terms by Wu & Yang (1968).

As we will see, the key point is to remove a point (the position of the monopole) from space $\mathbb{R}^3$.

Then $\nabla \cdot \underline{B} = 0$ everywhere in $\mathbb{R}^3 \setminus \overset{\text{origin}}{0}$, but we can have

$$m = \frac{1}{2\pi} \int_{S^2} \underline{B} \cdot \underbrace{d\underline{\sigma}}_{\text{area element}} \neq 0 \qquad \text{MAGNETIC CHARGE} \tag{8.45}$$



magnetic flux through the 2-sphere surrounding the location of the monopole.

- REMARK:

(8.45) can also be phrased as

$$\underline{\nabla} \cdot \underline{B} = 2\pi m \, \delta^{(3)}(\underline{x}) \qquad \text{in } \mathbb{R}^3 \tag{8.46}$$

using Gauss' theorem (but it's better to work in $\mathbb{R}^3 \setminus 0$).

Proof: * Ex 38.

A2 YES! DIRAC MONOPOLE (AND DIRAC STRINGS)

- Using polar coordinates in $\mathbb{R}^3$, we have

$$\Delta \frac{1}{r} = -4\pi \delta^3(\underline{x}), \qquad \nabla \frac{1}{r} = -\frac{\underline{x}}{r^3} \tag{8.47}$$

where $r = |\underline{x}|$. Then we can solve (8.46) by

$$\underline{B} = \frac{m}{2} \frac{\underline{x}}{r^3} = \frac{m}{2} \frac{1}{r^2} \hat{x}, \tag{8.48}$$

similarly to how we describe pointlike electric charges.

- What about the vector potential/gauge field $\underline{A}$? Consider

Consider $\underline{A}^{\overset{\text{label}}{+}}$ given by

$$A_x^+ = -\frac{m}{2} \frac{y}{r(r+z)}, \qquad A_y^+ = \frac{m}{2} \frac{x}{r(r+z)}, \qquad A_z^+ = 0, \tag{8.49}$$

which has

$$\nabla \times \underline{A}^{+} = \frac{m}{2}\frac{\underline{x}}{r^3} + 2\pi m\, \delta(x)\delta(y)\,\theta(-z) \,. \qquad (8.50)$$

Proof : * Ex 38.

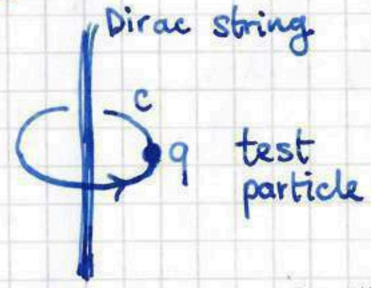
So we have

$$\underline{B}^{+} = \nabla \times \underline{A}^{+} = \underline{B} \qquad (8.51)$$

on $\mathbb{R}^3 \setminus \{\text{negative z-axis}\}$, where $\underline{A}^{+}$ is defined.

- The singularity along the -ve z-axis is called a DIRAC STRING. At first sight it looks problematic, but Dirac realized that it's not observable if the magnetic charge is properly quantized : $m \in \mathbb{Z}$.

Understanding this properly requires QM, but we can also see this as follows: consider moving a test particle of electric charge $q \in \mathbb{Z}$ slowly around the Dirac string.



Its quantum wavefunction is multiplied by picks up a phase from parallel transport :

$$e^{iq\oint_C \underline{A}\cdot d\underline{\ell}} = W_C^{(q)} \,. \leftarrow \begin{pmatrix}\text{Wilson loop along } C \\ \text{in rep of charge } q\end{pmatrix} \qquad (8.52)$$

dx

One can show that this phase is $=1$, i.e. unobservable, $\forall q \in \mathbb{Z}$ iff $m \in \mathbb{Z}$.

Proof : * Ex 38.

• In fact, there's no need to worry about Dirac strings, as was understood by Wu & Yang (1968).

KEY IDEA: gauge fields are defined locally (connection of fibre bundle).

- Think of $\mathbb{R}^3$ as $\mathbb{R}_+ \times S^2$ (polar coordinate) and cover S^2 by 2 patches U_+ and U_-. In addition to $\underline{A}^+$ defined on U_+,

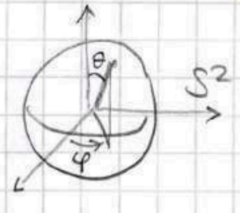
$$A^-_x = \frac{m}{2}\frac{y}{r(r-z)}, \quad A^-_y = -\frac{m}{2}\frac{x}{r(r-z)}, \quad A^-_z = 0 \qquad (8.53)$$

defined on the southern patch U_-. (which can be taken to be $S^2 \setminus N$)

- $\underline{A}^+, \underline{A}^-$ are non-singular where they are defined, and are related by a $U(1)$ gauge transfo on the overlap $U_+ \cap U_-$:

$$A^\pm = A^\pm_x\, dx + A^\pm_y\, dy + A^\pm_z\, dz = A^\pm_r\, dr + A^\pm_\theta\, d\theta + A^\pm_\varphi\, d\varphi,$$

$$A^\pm = \frac{m}{2}(\pm 1 - \cos\theta)\, d\varphi \qquad (8.54)$$



(* Ex 39), so

$$A^+ - A^- = m\, d\varphi = d(m\varphi) \equiv d\alpha_{+-}$$
$$\equiv -i g^{-1}_{+-}\, d g_{+-} \qquad (8.55)$$

where the transition function (or gauge transfo parameter)

$$g_{+-}(\varphi) = e^{i\alpha_{+-}(\varphi)} = e^{im\varphi} \quad \in \underline{U(1)} \qquad (8.56)$$

Since $\varphi \sim \varphi + 2\pi$ (angle, e.g. along the equator), g_{+-} is single-valued as we do one lap around the equator iff $m \in \mathbb{Z}$:

$$g_{+-}(\varphi + 2\pi) = g_{+-}(\varphi) \iff \underline{m \in \mathbb{Z}}. \qquad (8.57)$$

So the quantization of the magnetic charge follows from requiring that $A^\pm$ are connections for a $U(1)$ principal bundle over S^2. (The charged test particle in Dirac's argument is then a section of an associated line bundle.)

• REMARKS :

1) In this formulation, calling $U_{N \atop S} = {\text{northern} \atop \text{southern}}$ hemisphere,

opposite orientations

$$
\begin{aligned}
\frac{1}{2\pi}\,\Phi_{S^2}(\underline{B}) &= \frac{1}{2\pi}\int_{S^2}\underline{B}\cdot d\underline{\sigma} = \frac{1}{2\pi}\int_{U_N}\underline{B}^+\cdot d\underline{\sigma} + \frac{1}{2\pi}\int_{U_S}\underline{B}^-\cdot d\underline{\sigma} \\
&= \frac{1}{2\pi}\int_{U_N}(\nabla\times\underline{A}^+)\cdot d\underline{\sigma} + \frac{1}{2\pi}\int_{U_S}(\nabla\times\underline{A}^-)\cdot d\underline{\sigma} \\
&= \frac{1}{2\pi}\oint_{\text{equator}}\underline{A}^+\cdot d\underline{\ell} - \frac{1}{2\pi}\oint_{\text{equator}}\underline{A}^-\cdot d\underline{\ell} \\
&= \frac{1}{2\pi}\oint_{\text{equator}}(\underline{A}^+-\underline{A}^-) = \frac{m}{2\pi}\int_0^{2\pi}\underbrace{d\varphi}_{= d\alpha_{+-}} = m\,.
\end{aligned}
\tag{8.58}
$$

2) If $\underline{m=1}$, the relevant principal $U(1)$ bundle on S^2 is the HOPF BUNDLE of Ex 27 (total space $E = S^3$).

• Very nice! We can describe a static solution of Maxwell's eqn which is a pointlike magnetic charge (magnetic monopole) by excising the location of the monopole from space and exploiting geometry and topology of bundles over $\mathbb{R}^3 \setminus$ point (or equivalently S^2).

BUT (* Ex 39) :

(Show that) a Dirac monopole has infinite energy!

So $\boxed{A3}$ No!(?)

We'll fix this next.

8.5 - The 't Hooft - Polyakov monopole

- In 1974 't Hooft and Polyakov independently discovered that non-abelian gauge theories with scalars in the adjoint rep admit smooth monopoles as static finite energy solutions of the EoM.

- <u>GEORGI – GLASHOW MODEL</u> (SU(2) ADJOINT HIGGS MODEL)

Gauge group $G = SU(2)$, $\Phi \in$ <u>adj</u> (triplet, 3-dim'l irrep) view as <u>2x2 traceless hermitian</u>

$$\mathcal{L} = -\frac{1}{2g_{YM}^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \mathrm{tr}((D_\mu\phi)(D^\mu\phi)) - V(\phi),$$
$$V(\phi) = \lambda\left(\frac{1}{2}\mathrm{tr}(\phi^2) - v^2\right)^2, \qquad (v>0) \qquad (8.59)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$
$$D_\mu\Phi = \partial_\mu\Phi - i[A_\mu, \Phi]\,. \qquad (8.60)$$

- The <u>ENERGY</u> of <u>STATIC</u> field configurations is (gauge $A_0 = 0$)

$$E = \int d^3x\left[\frac{1}{g_{YM}^2}\,\mathrm{tr}(\overbrace{\underbrace{B_i}_{=-\frac{1}{2}\varepsilon_{ijk}F_{jk}}B_i}^{=\underline{B}^2}) + \mathrm{tr}(\overbrace{(D_i\phi)(D_i\phi)}^{=(\underline{D}\phi)^2}) + V(\Phi)\right]. \qquad (8.61)$$

We can minimize E by setting

$$\underline{B} = 0, \quad \underline{D}\Phi = 0, \quad \mathrm{tr}(\phi^2) = 2v^2\,. \qquad (8.62)$$

$\underline{B}=0 \Leftrightarrow F_{ij} = 0 \Leftrightarrow A_i$ is pure gauge

$\underline{D}\Phi = 0 \Leftrightarrow \phi$ covariantly constant

By a gauge transfo we can set $\underline{A} = 0$, so Φ is <u>constant</u> and

$$\mathrm{tr}(\phi^2) = 2v^2 \underset{\phi=\phi_a\sigma_a}{\Longleftrightarrow} \phi_1^2 + \phi_2^2 + \phi_3^2 = v^2\,. \qquad (8.63)$$

Proof: <u>* Ex 40</u>

So the <u>vacuum manifold</u> is

$$\mathbf{V} = \{\phi \mid \mathrm{tr}(\phi^2) = v^2\} \cong S^2 . \quad \text{(radius} = v) \qquad (8.64)$$

By a further (constant) gauge transfo, we can take

$$\Phi = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} = v\,\sigma_3 , \qquad (8.65)$$

which breaks ("spontaneously" in physics parlance) $G = SU(2)$ to $H = U(1)$.

$\mathcal{L}$ is invariant
Ground state is not invariant

Proof: * <u>Ex 40</u> .

- In order for <u>the energy</u> (8.61) <u>to be finite</u>, we require

<u>BC</u>: $\quad \underline{B} \to 0, \quad \underline{D}\phi \to 0, \quad \mathrm{tr}(\phi^2) \to 2v^2 \quad \text{as } |\vec{x}| \to \infty \qquad (8.66)$

so the fields must tend to a vacuum at spatial infinity.
(this can be a different vacuum for each direction!)
So at spatial infinity we have a map

$$\Phi_\infty : S^2_\infty \longrightarrow V \cong S^2 , \qquad (8.67)$$

which is characterized by an <u>integer</u>, the <u>TOPOLOGICAL DEGREE of the map</u>, which is a generalization of the winding number:
(Mathematically, this is because $\pi_2(S^2) = \mathbb{Z}$.)

$$\nu = \frac{1}{4\pi v^3} \int_{S^2_\infty} \frac{1}{2} \varepsilon_{ijk}\, \vec{\phi}_\infty \cdot \left(\partial_j \vec{\phi}_\infty \times \partial_k \vec{\phi}_\infty\right) d\sigma_i \qquad (8.68)$$

where $\vec{\phi}_\infty = ((\phi_\infty)_1, (\phi_\infty)_2, (\phi_\infty)_3)$.

* <u>Ex</u> : write the integrand of (8.68) in terms of the <u>matrix</u> $\phi_\infty = (\phi_\infty)_a \sigma_a$.

*** Ex 41**: Let

$$F^{\overset{H}{\overset{\parallel}{U(1)}}}_{\mu\nu} := \frac{1}{2v}\, \mathrm{tr}\left(\Phi_\infty F_{\mu\nu}\right) \tag{8.69}$$

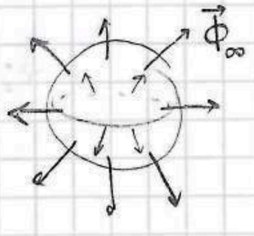
be the field strength for the unbroken $\overset{H}{\overset{\parallel}{U(1)}} \subset \overset{G}{\overset{\parallel}{SU(2)}}$. Show that the corresponding <u>magnetic charge</u>

$$m^{U(1)} := \frac{1}{2\pi} \int_{S^2_\infty} \underline{B}^{U(1)} \cdot d\underline{\sigma} \tag{8.70}$$

is proportional to the degree ν of ϕ_∞, with a proportionality factor that you should find.

<u>EXAMPLE</u>:

$$\phi_\infty = v\, \hat{r} \cdot \underline{\sigma} \tag{8.71}$$



has <u>$\nu = 1$</u>. We can write

$$\phi_\infty = v\, e^{-i\alpha} \sigma_3 e^{i\alpha} \tag{8.72}$$

with

$$\alpha = \frac{\theta}{2}\left(-\sigma_1 \sin\varphi + \sigma_2 \cos\varphi\right) = \frac{\theta}{2} e^{i\varphi\sigma_3} \tag{8.73}$$

a gauge transformation parameter which is singular at $\theta = \pi$.
(Polar coord's (r, θ, φ) in $\mathbb{R}^3$)

• <u>'t Hooft – Polyakov "hedgehog" ansatz</u>:

$$\phi = \frac{\underline{x} \cdot \underline{\sigma}}{r^2} H(vr)$$
$$A = \sigma_a \varepsilon_{aij} \frac{x_i dx_j}{r^2} \left(1 - K(ar)\right) \tag{8.74}$$
(up to factors of 2 possibly...)

with

$$\underline{\xi \equiv vr \to \infty}: \quad H(\xi) - \xi \to 0, \quad K(\xi) \to 0 \quad \left(\begin{smallmatrix}\text{fast}\\ \text{enough}\end{smallmatrix}\right) \tag{8.75}$$

to ensure <u>finite energy</u>, and

Solution looks like a Dirac monopole for $H = U(1)$ as $\xi \to \infty$!

$\underline{\xi \to 0}$: $\quad H = O(\xi), \quad K-1 = O(\xi)$ (8.76)

to ensure regularity (smoothness) at the centre of the monopole.

• BOGOMOL'NYI (-PRASAD-SOMMERFIELD) BOUND: for static fields w/ $A_0 = c$

$$E = \int d^3x \left[\frac{1}{g_{YM}^2} \mathrm{tr}(\underline{B}^2) + \mathrm{tr}((\underline{D}\phi)^2) + \underbrace{V(\phi)}_{\geq 0} \right]$$

$$\geq \int d^3x \left[\mathrm{tr}\left(\underbrace{\left(\frac{1}{g_{YM}} \underline{B} \mp \underline{D}\phi\right)^2}_{\geq 0} \pm \frac{2}{g_{YM}} \underline{B}\cdot\underline{D}\phi \right)\right]$$

$$\geq \pm \frac{2}{g_{YM}} \int d^3x \, \mathrm{tr}(\underline{B}\cdot\underline{D}\phi)$$

$$\boxed{\begin{matrix}\text{Bianchi id.}\\ \underline{D}\cdot\underline{B}=0\end{matrix}} \; = \pm \frac{2}{g_{YM}} \int d^3x \, \mathrm{tr}(\underline{D}\cdot(\underline{B}\phi)) \qquad (8.77)$$

$$\boxed{\begin{matrix}\mathrm{tr}(\underline{D}\cdot\ldots)\\ = \nabla\cdot\mathrm{tr}(\ldots)\\ \text{since tr is gauge}\\ \text{invariant}\end{matrix}} \; = \pm \frac{2}{g_{YM}} \int d^3x \, \nabla\cdot\mathrm{tr}(\phi\underline{B})$$

$$\underset{\text{Gauss}}{=} \pm \frac{2}{g_{YM}} \int_{S^2_\infty} \mathrm{tr}(\phi_\infty \underline{B})\cdot d\underline{\sigma}$$

$$= \pm \frac{4v}{g_{YM}} \int_{S^2_\infty} \underline{B}^{U(1)}\cdot d\underline{\sigma} = \pm \frac{8\pi v}{g_{YM}} m^{U(1)}$$

So

$$\underbrace{E}_{\text{energy}} \geq \frac{8\pi v}{g_{YM}} \underbrace{|m^{U(1)}|}_{\text{magnetic charge}} \qquad \text{BPS BOUND} \qquad (8.78)$$

The bound is saturated $\left(E = \frac{8\pi v}{g_{YM}} |m^{U(1)}| \right)$ for static configurations iff:

$\lambda \to 0$ keeping v fixed "BPS LIMIT"

$\underline{B} = \underbrace{\pm}_{\mathrm{sign}(m^{U(1)})} g_{YM} \underline{D}\phi$. BPS/BOGOMOL'NYI EQN (8.79)

(1st order, easier to solve)