Geometry of Mathematical Physics III (Epiphany 2023)

Exercises

April 25, 2023

5 Abelian gauge theories

Ex 1

1. Given $F^{\mu\nu}$ in equation (5.7) of the notes, show that $F_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}F^{\rho\sigma}$ are the components of the matrix in equation (5.8) of the notes.
2. Find $\epsilon_{\mu'\nu'\rho'\sigma'} = \eta_{\mu\mu'}\eta_{\nu\nu'}\eta_{\rho\rho'}\eta_{\sigma\sigma'}\epsilon^{\mu\nu\rho\sigma}$, where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric rank 4 tensor (*i.e.*tensor with 4 indices) normalized such that $\epsilon^{0123} = 1$.
3. Starting from the non-relativistic form of the Maxwell equations (5.1) in the notes, the field strength $F^{\mu\nu}$ in equation (5.7) of the notes and the 4-current $(J^\mu) = (\rho, \boldsymbol{j})$, show that the Maxwell equations can be rewritten in the relativistic form
$$\partial_\nu F^{\mu\nu} = J^\mu \ , \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \ .$$
4. Write $F^{\mu\nu}F_{\mu\nu}$ in terms of electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$.
5. Using the transformation properties of the tensors J^μ and $F^{\mu\nu}$, work out how ρ, $\boldsymbol{j}$, $\boldsymbol{E}$ and $\boldsymbol{B}$ transform under a Lorentz boost
$$\Lambda = (\Lambda^\mu{}_\nu) = \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .$$

Ex 2 Let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

1. Using the expression of $F_{\mu\nu}$ in terms of electric and magnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$ and $(A^\mu) = (\phi, \boldsymbol{A})$, show that
$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \ , \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} \ .$$
2. Write the relativistic form of the Maxwell equations in terms of A_μ, and then express them in terms of ϕ and $\boldsymbol{A}$.

Ex 3 Consider two independent rank-n tensors X and Y in d spacetime dimensions.

1. Show that
$$\frac{\partial}{\partial X_{a_1\ldots a_n}}(X^{b_1\ldots b_n}Y_{b_1\ldots b_n}) = \frac{\partial}{\partial X_{a_1\ldots a_n}}(X_{b_1\ldots b_n}Y^{b_1\ldots b_n}) = Y^{a_1a_2\ldots a_n}$$
and
$$\frac{\partial}{\partial X_{a_1\ldots a_n}}(X^{b_1\ldots b_n}X_{b_1\ldots b_n}) = 2X^{a_1a_2\ldots a_n} \ .$$

2. Now set $n = 3$. Show that
$$\frac{\partial}{\partial X_{abc}} X_{abe} = d^2 \delta_c^e \ .$$

Ex 4 Let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

1. Show that the vacuum Maxwell equations, which are obtained by setting $J^\mu = 0$, have plane wave solutions of the form
$$A_\mu = \mathrm{Re}(p_\mu e^{ik_\nu x^\nu})$$
for constant vectors p_μ and k_μ. Which conditions must p_μ and k_μ obey?
2. Taking into account the freedom of gauge transformations, how many physically independent components of p_μ appear in a general plane wave solution? [**Hint**: try plane waves $\alpha = \mathrm{Re}(ce^{ik_\nu x^\nu})$.]

Ex 5

1. Work out the components of
$$\tilde{F}^{\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$
and of $\tilde{F}_{\mu\nu}$ in terms of the electric and magnetic field.
2. Find the solutions to the equations $F_{\mu\nu} = \pm i \tilde{F}_{\mu\nu}$ in terms of electric and magnetic field, and count the number of degrees of freedom that they contain. Which representations of the Lorentz group do they correspond to? Work out how the components transform under rotations and boosts.

Ex 6 Consider a complex scalar field ϕ with Lagrangian density
$$\mathcal{L} = -|\partial_\mu \phi|^2 - V(\phi, \bar{\phi}) = -|\partial_\mu \phi|^2 - \lambda(|\phi|^2 - a^2)^2 \ ,$$
with parameters $\lambda, a > 0$.

1. Show that the energy (or "Hamiltonian") is[1]
$$E = \int d^3x \ \left(|\partial_0 \phi|^2 + |\partial_i \phi|^2 + V(\phi, \bar{\phi}) \right) \ .$$

[1] You may use the relation between the Lagrangian and Hamiltonian densities
$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi} \partial_0 \phi + \frac{\partial \mathcal{L}}{\partial \partial_0 \bar{\phi}} \partial_0 \bar{\phi} - \mathcal{L}$$
that you learned in Mathematical Physics. Alternatively, you may use that E is the Noether charge associated to invariance under time translations $t \mapsto t + c$.

2. Find the configurations of least energy ("vacua", or "ground states") and show that they parametrize a circle in field space.
3. Show that any two vacua are related by a global $U(1)$ transformation.

Ex 7 Show that the commutator of two covariant derivatives $D_\mu = \partial_\mu - iA_\mu$ is given by

$$[D_\mu, D_\nu] = -iF_{\mu\nu} = -i(\partial_\mu A_\nu - \partial_\nu A_\mu) \ .$$

[**Hint**: start by calculating the commutators $[\partial_\mu, \partial_\nu]$, $[\partial_\mu, g]$ and $[g, h]$, where g and h are functions of x. If you are confused, apply the above commutators to smooth test functions $f(x)$.]

Ex 8 The scalar field ϕ in chapter 5 was assumed for simplicity to have charge 1.

1. Go through the chapter and adjust formulae where necessary to generalize to a complex scalar field ϕ of charge $q \in \mathbb{Z}$.
2. Let ϕ be a charge q field and $D^{(q)}_\mu$ be the covariant derivative that acts on it. Check that for its complex conjugate $\bar{\phi}$, which has the opposite charge $-q$, the covariant derivative satisfies
$$\overline{D^{(q)}_\mu \phi} = D^{(-q)}_\mu \bar{\phi} \ .$$

Ex 9 Write down the most general real gauge invariant Lagrangian with at most two derivatives for two complex scalar fields, ϕ of charge 1 and χ of charge 2, and a $U(1)$ gauge field A_μ, which comprises:

1. kinetic terms for ϕ and χ;
2. a kinetic term for A_μ;
3. a real gauge invariant potential which is a polynomial of degree at most 4 in ϕ, χ and their complex conjugates.

Ex 10 Consider "scalar electrodynamics", the field theory with Lagrangian density

$$\mathcal{L} = -\overline{D_\mu \phi} D^\mu \phi - U(|\phi|^2) - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \ , \tag{5.1}$$

where

$$D_\mu \phi = (\partial_\mu - iA_\mu)\phi \ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

1. Show that the equations of motion (Euler-Lagrange equations) for the complex scalar field ϕ and for the real $U(1)$ gauge field A_μ are
$$D_\mu D^\mu \phi = U'(|\phi|^2)\phi \ , \qquad \partial_\nu F^{\mu\nu} = g^2 J^\mu \ ,$$
where
$$J_\mu = -i(\bar{\phi} D_\mu \phi - \phi \overline{D_\mu \phi}) \ .$$
2. Show that the current J_μ is real, gauge invariant, and conserved ($\partial_\mu J^\mu = 0$) upon using the equations of motion.
3. Show that the conserved current J_μ in the gauge theory and the conserved current j_μ in the scalar field theory with $U(1)$ global symmetry, which is obtained by setting $A_\mu = 0$ in the Lagrangian (5.1), are related by
$$J_\mu = j_\mu + b A_\mu |\phi|^2 \ ,$$
for a constant b that you should find.

Ex 11 Consider scalar electrodynamics, the field theory introduced in Ex 10.

1. Show that it is always possible to (partially) fix a gauge where $A_0 = 0$.
[**Hint**: show that if $A_0 \neq 0$, then one can perform a suitable gauge transformation so that $A'_0 = 0$. Find this gauge transformation explicitly.]
2. * The previous argument is too fast. Where can it fail?
[**Note**: * signals advanced optional questions. I include them for you to think about them, but don't worry if you don't know how to answer them.]
3. Working in the gauge $A_0 = 0$, calculate the Hamiltonian density and show that the total energy (or Hamiltonian) is
$$E = \int d^3x \ \left(|\partial_0 \phi|^2 + |(\nabla - i\vec{A})\phi|^2 + U(|\phi|^2) + \frac{1}{2g^2}(\vec{E}^2 + \vec{B}^2) \right) \ ,$$
where $E_i = \partial_0 A_i$ and $B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$ are the electric and magnetic field respectively.

Ex 12 Consider the $U(1)$ Wilson loop (of charge 1) associated to the perimeter C of an infinitesimal rectangle in the (x^1, x^2) plane. Let the vertices of the rectangle be
$$(x^1, x^2) \ , \quad (x^1 + \epsilon dx^1, x^2) \ , \quad (x^1 + \epsilon dx^1, x^2 + \epsilon dx^2) \ , \quad (x^1, x^2 + \epsilon dx^2) \ ,$$
where the increments $\epsilon dx^1, \epsilon dx^2$ are infinitesimal ($\epsilon \ll 1$).

1. Write the Wilson loop $W_C = \exp\left[i \oint_C A_\mu dx^\mu\right]$ in terms of the Wilson lines associated to the four sides of the rectangle.
2. Calculate the Taylor expansion of the above Wilson loop up to and including the order ϵ^2, and express it in terms of known quantities.

Ex 13 Consider a $U(1)$ gauge field A_μ, with field strength $F_{\mu\nu}$ and dual field strength $\tilde{F}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$.

1. Write the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_\theta = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}$$

in terms of the electric and magnetic field. θ is a constant parameter, called the theta angle.

2. Show that

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \partial_\mu \omega^\mu \ ,$$

where ω^μ is a 4-vector that you should find. Then show that the equations of motion for the gauge field A_μ with the action $S = \int d^4x\ \mathcal{L}$ are independent of the theta angle.

6 Applications of abelian gauge theories

Ex 14 Consider a complex scalar field ϕ with Lagrangian density

$$\mathcal{L} = -|\partial_\mu \phi|^2 - U(|\phi|^2) = -|\partial_\mu \phi|^2 - \frac{\lambda}{2}(|\phi|^2 - v^2)^2 ,$$

with parameters $\lambda, v > 0$. For the purpose of this exercise, you can take for granted that the energy of field configurations is

$$E = \int d^2x \left[|\dot{\phi}|^2 + |\nabla \phi|^2 + U(|\phi|^2) \right] .$$

Working with static field configurations (that is, $\dot{\phi} \equiv \partial_t \phi = 0$), show that field configurations which extremize the energy obey the Euler-Lagrange equations and vice versa.

Ex 15 Let $\vec{x} = (x^1, x^2) \in \mathbb{R}^2$ and $\phi(\vec{x}) \in \mathbb{C}$ a complex scalar field varying over two-dimensional space. Identify space $\mathbb{R}^2 \cong \mathbb{C}$, with complex coordinate $z = x^1 + ix^2$. Associate to any loop (or closed curve) C in space the vorticity (or vortex number, or winding number)

$$N[C] = \frac{1}{2\pi} \oint_C \nabla \arg(\phi) \cdot d\vec{l} \equiv \frac{1}{2\pi} \oint_C \partial_i \arg(\phi)\; dx^i .$$

1. Parametrize the loop C by a periodic coordinate $\tau \sim \tau + 2\pi$, with $\tau = 0$ at the start point of the loop, and $\tau = 2\pi$ at the endpoint (which coincide with the start point.) Using that ϕ is a single-valued function, show that $N[C]$ is an integer as long as $\arg(\phi)$ is well-defined along C.
2. Use Stokes' theorem to write the vortex number as a surface integral, and show that it can only receive contributions from points where $\phi = 0$ or $1/\phi = 0$ (you can assume that ϕ is smooth otherwise).
3. Assume for simplicity that ϕ is a holomorphic function of z. Let z_0 be a zero of ϕ of order n if $n > 0$, and a pole of order $|n|$ if $n < 0$, that is $\phi(z) \approx c(z - z_0)^n$ near $z = z_0$, where $c \neq 0$. Show that $N[C_{z_0}] = n$ for any infinitesimal loop C_{z_0} which encircles z_0 counterclockwise.
4. Generalize the previous calculation to $\phi(z, \bar{z}) \approx c(z - z_0)^n \overline{(z - z_0)}^m$.

Ex 16 A static complex field $\phi(\vec{x})$ which varies in two space dimensions obeys the field equation

$$\nabla^2 \phi - \lambda(|\phi|^2 - v^2)\phi = 0 .$$

Let $z = x^1 + ix^2 = re^{i\theta}$, where $\vec{x} = (x^1, x^2)$ are real Cartesian coordinates, and (r, θ) are real polar coordinates, with $r \geq 0$ and $\theta \sim \theta + 2\pi$.

1. Show that the ansatz $\phi(\vec{x}) = f(r)e^{i\theta}$, where $f(r) \in \mathbb{R}$, has total vorticity $N \equiv N[S^1_\infty] = 1$.
2. Show that $f(r)$ obeys the ODE

$$f'' + \frac{1}{r}f' - \frac{1}{r^2}f + \lambda(v^2 - f^2)f = 0 \ ,$$

where primes denote derivatives with respect to the radial coordinate r.

Ex 17 A static field $\phi(\vec{x})$ in two space dimensions has energy

$$E = \int d^2x \left[|\nabla\phi|^2 + \frac{\lambda}{2}(|\phi|^2 - v^2)^2\right] \equiv \int d^2x \ \mathcal{E} \ .$$

1. Let $\phi(\vec{x}) = \rho(\vec{x})\exp(i\alpha(\vec{x}))$, where $\rho(\vec{x}), \alpha(\vec{x})$ are real functions. Show that

$$E = \int d^2x \left[(\nabla\rho)^2 + \rho^2(\nabla\alpha)^2 + +\frac{\lambda}{2}(\rho^2 - v^2)^2\right] \ .$$

2. Let $\phi = f(r)\exp(i\theta)$ where (r, θ) are polar coordinates on the spatial plane, and $f(r)$ is a real function which obeys the boundary conditions $f(0) = 0$ and $f(\infty) = v$. Show that

$$\rho^2(\nabla\alpha)^2 = \frac{f^2}{r^2}$$

and use the boundary conditions to show that this causes a logarithmic divergence of the energy E as $r \to \infty$. That is, let

$$E_R \equiv \int_{r\leq R} d^2x \ \mathcal{E}$$

and show that $E_R \sim \log R$ as $R \to \infty$.

Ex 18 A scalar field theory in D space and 1 time dimensions has Lagrangian density

$$\mathcal{L} = -\frac{1}{2}G_{ab}(\phi)\partial_\mu\phi^a\partial_\mu\phi^b - V(\phi) \ ,$$

where $\mu, \nu = 0, 1, 2, \ldots, D$ and $x = (x^\mu) = (x^0, \vec{x})$. The matrix $G_{ab}(\phi)$ is positive definite, and the scalar potential is assumed to be non-negative: $V(\phi) \geq 0$.

The energy of static field configurations $\phi(\vec{x})$, which obey $\partial_0\phi(x) = 0$, is given by

$$\begin{aligned} E[\phi] &= E_K[\phi] + E_V[\phi] \ , \\ E_K[\phi] &= \int d^Dx \ \frac{1}{2}G_{ab}(\phi)\partial_i\phi^a\partial_i\phi^b \\ E_V[\phi] &= \int d^Dx \ V(\phi) \ . \end{aligned}$$

1. Show that any finite energy static solution of the Euler-Lagrange field equations is a stationary point of the static energy $E[\phi]$.
2. Let $\phi(\vec{x}) = \phi_1(\vec{x})$ be a static solution of the field equations, and consider the one-parameter family of field configurations

$$\phi(\vec{x}) = \phi_\lambda(\vec{x}) := \phi_1(\lambda\vec{x})$$

labelled by the parameter $\lambda > 0$. Show that

$$E[\phi_\lambda] = E_K[\phi_\lambda] + E_V[\phi_\lambda] = \lambda^{2-D} E_K[\phi_1] + \lambda^{-D} E_V[\phi_1] \ .$$

Ex 19 The Abelian Higgs model in 2 space and 1 time dimensions has Lagrangian density

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} - \overline{D_\mu\phi} D^\mu \phi - \frac{\lambda}{2}(|\phi^2| - v^2)^2 \ ,$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \qquad D_\mu\phi = \partial_\mu\phi - iA_\mu\phi \ ,$$

and g, λ, v are positive constants.

1. Find the equations of motion for ϕ and A_μ.
2. Work in the gauge $A_0 = 0$, and consider static solutions of the form

$$\phi(\vec{x}) = ve^{i\theta} f(vr)$$
$$A_j(\vec{x}) = \epsilon_{jk}\hat{x}^k \frac{a(vr)}{r} \ ,$$

where (r, θ) are polar coordinates on the spatial plane, that is $x^1 = r\cos\theta$, $x^2 = r\sin\theta$. Show that under the above ansatz the equations of motion reduce to a system of two ODEs for $f(r)$ and $a(r)$, that you should find.

Ex 20 Let $\phi^{1,2}(x)$ be two real scalar fields in two space and one time dimensions (x^0, x^1, x^2), and $\phi(x) = \phi^1(x) + i\phi^2(x)$.

1. Show that the current

$$j^\mu = c\frac{1}{2}\epsilon_{ab}\epsilon^{\mu\nu\rho}\partial_\nu(\phi^a\partial_\rho\phi^b)$$

is conserved, that is $\partial_\mu j^\mu = 0$, regardless of the equations of motion. Here c is a normalization constant, $\epsilon^{\mu\nu\rho}$ is the totally antisymmetric tensor in three indices with $\epsilon^{012} = 1$, and ϵ_{ab} is the totally antisymmetric tensor in two indices with $\epsilon_{12} = 1$.

2. Assume that $|\phi| \to v$ at spatial infinity, where v is a constant. Write down the conserved charge Q associated to the current j^μ, and show that Q is equal to the total winding number of the argument of ϕ,

$$N = \frac{1}{2\pi} \int_{S^1_\infty} \nabla \arg(\phi) \cdot d\vec{l} \, ,$$

for a suitable choice of the normalization constant c that you should find.

3. Let $z = x^1 + ix^2$ be a complex coordinate on the spatial plane. For each choice of the sign $\epsilon = \pm 1$, rewrite the Bogomol'nyi equations for the abelian Higgs model

$$(D_1 - i\epsilon D_2)\phi = 0$$
$$F_{12} = \epsilon g^2(|\phi|^2 - v^2)$$

in terms of the complex coordinates $(z, \bar{z})$ rather than (x^1, x^2). Solve the first Bogomol'nyi equation to determine the holomorphic and antiholomorphic components $A_z, A_{\bar{z}}$ of the gauge field (remember that A_μ is real if $\mu = x^1, x^2$). Substitute the result in the second Bogomol'nyi equation to obtain a partial differential equation for $|\phi|^2$ only.

Ex 21 A magnetic monopole of magnetic charge m located at the origin O of three-dimensional space is described by a divergence-free magnetic field $\vec{B}$ in $\mathbb{R}^3 \setminus O$, with non-vanishing magnetic flux though the 2-sphere that surrounds the origin O:

$$\frac{1}{2\pi} \int_{S^2} \vec{B} \cdot d\vec{\sigma} = m \neq 0.$$

1. Show that all of the above can be reformulated as the equation

$$\nabla \cdot \vec{B} = 2\pi m \; \delta^{(3)}(\vec{x})$$

in $\mathbb{R}^3$.

2. Using that

$$\nabla \frac{1}{r} = -\frac{\vec{x}}{r^3} \, , \qquad \Delta \frac{1}{r} = -4\pi \; \delta^{(3)}(\vec{x}) \, ,$$

where $r = |\vec{x}|$ and $\Delta = \nabla^2$ is the Laplacian, show that

$$\vec{B} = \frac{m}{2} \frac{\vec{x}}{r^3}$$

solves the equation in part 1.

3. Show that the vector potentials $\vec{A}^{\pm}$ for a Dirac monopole, with components

$$A_x^{\pm} = \mp\frac{m}{2}\frac{y}{r(r\pm z)} \ , \quad A_y^{\pm} = \pm\frac{m}{2}\frac{x}{r(r\pm z)} \ , \quad A_z^{\pm} = 0$$

satisfy the equations

$$\nabla\times\vec{A}^{\pm} = \frac{m}{2}\frac{\vec{x}}{r^3}$$

in the regions where they are defined.

Ex 22 In the formulation of Wu and Yang, the gauge field of a Dirac monopole is described in two patches for $\mathbb{R}_+\times S^2$, which are given by $\mathbb{R}_+\times U_+$ and $\mathbb{R}_+\times U_-$, where U_+ is the region of S^2 north of the Southern tropic, and U_- is the region of S^2 south of the Northern tropic. The vector potentials in the two patches are

$$A_x^{+} = -\frac{m}{2}\frac{y}{r(r+z)} \ , \quad A_y^{+} = \frac{m}{2}\frac{x}{r(r+z)} \ , \quad A_z^{+} = 0$$

in the northern patch $\mathbb{R}_+\times U_+$, and

$$A_x^{-} = \frac{m}{2}\frac{y}{r(r-z)} \ , \quad A_y^{-} = -\frac{m}{2}\frac{x}{r(r-z)} \ , \quad A_z^{-} = 0$$

in the southern patch $\mathbb{R}_+\times U_-$.

1. Switching from cartesian coordinates (x,y,z) to polar coordinates (r,θ,φ) for $\mathbb{R}^3$, and using

$$A^{\pm} = A_x^{\pm}dx + A_y^{\pm}dy + A_z^{\pm}dz = A_r^{\pm}dr + A_\theta^{\pm}d\theta + A_\varphi^{\pm}d\varphi \ ,$$

find expressions for $A^{\pm}$ in polar coordinates. Show that[2]

$$F^{\pm} = dA^{\pm} = \frac{m}{2}\ \sin\theta\ d\theta\wedge d\varphi$$

and that on the overlap of the two patches

$$A^{+} - A^{-} = m\ d\varphi \ .$$

2. Show that the energy of a Dirac monopole

$$E = \frac{1}{2g^2}\int d^3x\ \vec{B}^2$$

is infinite, where $\vec{B} = \vec{B}^{\pm} := \nabla\times\vec{A}^{\pm}$ in the regions where the two vector potentials $\vec{A}^{\pm}$ are defined.

[2]This requires a part of the crash course on differential geometry which I skipped in the lecture, but please try if you are interested.

7 Non-abelian gauge theories

Ex 23

1. Consider a compact simple Lie algebra, with generators t_a, the Lie bracket
$$[t_a, t_b] = i f_{ab}{}^c t_c \ ,$$
and Killing form[3]
$$K_{ab} := \mathrm{tr}(t_a t_b) \ .$$
Use the antisymmetry of the Lie bracket to show that the structure constants with lowered indices, defined as
$$f_{abc} := f_{ab}{}^d K_{dc}$$
are completely antisymmetric in their indices:
$$f_{abc} = -f_{bac} = -f_{cba} \ .$$

2. Show that for any representation $\mathbf{r}$ of a simple Lie algebra
$$\mathrm{tr}_{\mathbf{r}}(t_a^{(\mathbf{r})}) = 0 \ ,$$
where $\mathrm{tr}_{\mathbf{r}}$ is the trace in the representation $\mathbf{r}$.

Ex 24

1. Consider a representation $\mathbf{r}$ of a compact Lie G, with generators $t_a^{(\mathbf{r})}$. Show that the set of generators
$$t_a^{(\bar{\mathbf{r}})} := -(t_a^{(\mathbf{r})})^T$$
defines another representation of G, which is called the complex conjugate representation $\bar{\mathbf{r}}$. (The subscript T denotes the transposition of matrices.)

2. Show that if the column vector ϕ transforms in the irrep $\mathbf{r}$, then its complex conjugate $\phi^* \equiv \bar{\phi}$, which is also a column vector, transforms in the complex conjugate irrep $\bar{\mathbf{r}}$.

3. Denote $\bar{\phi}_j := (\phi^j)^*$ and construct the row vector $\phi^\dagger = \bar{\phi}^T = (\bar{\phi}_1, \ldots, \bar{\phi}_r)$. Show that the inner product $\phi^\dagger\phi$ is invariant under the action of G.

[3]The Killing form as defined in the lecture notes is $K(v, w) = K_{ab} v^a w^b$. So K_{ab} is the matrix expression of the Killing form in the basis of generators.

Ex 25

1. The adjoint action of the Lie algebra $\mathfrak{g}$ on itself is given by

$$\begin{array}{rccl} \text{ad:} & \mathfrak{g} & \to & \mathfrak{g} \\ & y & \mapsto & \text{ad}_x(y) := [x, y] \end{array}$$

for all Lie algebra elements $x \in \mathfrak{g}$. Show that

$$\text{ad}_{t_a}(y^b t_b) = (t_a^{(\text{adj})})^b{}_c y^c t_b \ .$$

where

$$(t_a^{(\text{adj})})^b{}_c := i f_{ac}{}^b$$

are the generators in the adjoint representation.

2. Express the Killing form

$$K_{ab} = \text{tr}_{\text{adj}}(t_a^{(\text{adj})} t_b^{(\text{adj})})$$

in terms of the structure constants.

3. Show that in a basis where the Killing form is

$$K_{ab} = C(\text{adj})\delta_{ab}$$

the quadratic invariant $C(\text{adj})$ of the adjoint representation is given by

$$C(\text{adj}) = \frac{\delta^{ab} f_{acd} f_b{}^{cd}}{\dim \mathfrak{g}} \ .$$

Ex 26 Let $\text{Mat}_\text{n}(\text{F})$ denote the set of $n \times n$ matrices whose entries are in the field F, and 1_n the $n \times n$ identity matrix. The classical compact simple Lie groups are

$$\begin{aligned} SU(N) &= \{g \in \text{Mat}_\text{N}(\mathbb{C}) \mid \text{g}^\dagger \text{g} = 1_\text{N} \ , \quad \det \text{g} = 1\} \\ SO(N) &= \{g \in \text{Mat}_\text{N}(\mathbb{R}) \mid \text{g}^\text{T} \text{g} = 1_\text{N} \ , \quad \det \text{g} = 1\} \\ USp(2N) &= \{g \in \text{Mat}_{2\text{N}}(\mathbb{C}) \mid \text{g}^\dagger \text{g} = 1_{2\text{N}} \ , \quad \text{g}^\text{T} \text{J} \text{g} = \text{J}\} \end{aligned}$$

where the $(2N) \times (2N)$ antisymmetric matrix

$$J = \begin{pmatrix} 0_N & 1_N \\ -1_N & 0_N \end{pmatrix}$$

is called the symplectic form.

1. Characterize the Lie algebras $su(N)$, $so(N)$, and $usp(2N)$ as vector spaces of matrices subject to certain linear conditions, which you should find.
[**Hint**: You can assume that a group element takes the exponential form $g = \exp(i\alpha^a t_a)$ and Taylor expand for infinitesimal α.]

2. Find the generators of the fundamental representation fund and its complex conjugate rep $\overline{\text{fund}}$ (the so called antifundamental representation) for $G = SU(N)$, $SO(N)$, $USp(2N)$.
3. For $G = SO(N), USp(2N)$, show that fund and $\overline{\text{fund}}$ are equivalent representations, namely
$$t_a^{(\overline{\text{fund}})} = V t_a^{(\text{fund})} V^{-1} \qquad \forall a$$
for some invertible matrix V that you should find.

Ex 27 The Lie algebra $su(2)$ of the group $SU(2)$ has three generators t_1, t_2, t_3 and Lie brackets

$$[t_1, t_2] = it_3 \ , \quad [t_2, t_3] = it_1 \ , \quad [t_3, t_1] = it_2 \ ,$$

where we have fixed the normalization once and for all.

1. Write down all the structure constants $f_{ab}{}^c$ of the Lie algebra $su(2)$.
2. Write down generators $t_a^{(\mathbf{2})}$ for the doublet (2-dimensional, or fundamental) representation $\mathbf{2}$, in the above normalization. Calculate the trace $\mathrm{tr}_{\mathbf{2}}(t_a^{(\mathbf{2})} t_b^{(\mathbf{2})})$ and hence the quadratic invariant $C(\mathbf{2})$ of the doublet representation.
3. Write down generators $t_a^{(\mathbf{3})}$ for the triplet (3-dimensional, or adjoint) representation $\mathbf{3}$, in the above normalization. Calculate the trace $\mathrm{tr}_{\mathbf{3}}(t_a^{(\mathbf{3})} t_b^{(\mathbf{3})})$ and hence the quadratic invariant $C(\mathbf{3})$ of the triplet representation.

Ex 28 By considering infinitesimal gauge transformations ($|\alpha^a| \ll 1$)

$$g = e^{i\alpha^a t_a} \equiv e^{i\alpha} = 1 + i\alpha + O(\alpha^2)$$

and Taylor expanding finite gauge transformations to leading order in $\alpha \in \mathfrak{g} = \mathrm{Lie}(G)$, show that the **infinitesimal gauge variations** of the fields are

$$\begin{aligned} \delta_\alpha \phi &= i\alpha\phi \\ \delta_\alpha A_\mu &= i[\alpha, A_\mu] + \partial_\mu \alpha \\ \delta_\alpha F_{\mu\nu} &= i[\alpha, F_{\mu\nu}] \ , \end{aligned}$$

where $\phi \mapsto \phi + \delta_\alpha \phi + O(\alpha^2)$ and so on.

Ex 29 Specialize the equations written in section 7.2 of the lecture notes to the case of the gauge group $G = U(1)$, and show that they reduce to the equations written in chapter 5. Do it both for the charge 1 representation, which is analogous to the fundamental representation, and for the more general charge q representation.

Ex 30 Consider a field ϕ transforming in the adjoint representation adj, with components ϕ^a, where $a = 1, \ldots, \dim\ \mathfrak{g}$.

1. Show that
$$(A_\mu \phi)^a = i f_{bc}{}^a A^b_\mu \phi^c$$
and similarly for $(F_{\mu\nu}\phi)^a$.
2. Let $\Phi := \phi^a t_a$, and $A_\mu = A^a_\mu t_a$, $F_{\mu\nu} = F^a_{\mu\nu} t_a$ as usual. Show that
$$(A_\mu \phi)^a t_a = [A_\mu, \Phi]$$
and similarly for $F_{\mu\nu}\phi$. Show that therefore
$$D_\mu \Phi = \partial_\mu \Phi - i[A_\mu, \Phi]$$
$$[D_\mu, D_\nu]\Phi = -i[F_{\mu\nu}, \Phi] \ .$$

Ex 31 Consider a gauge group G, with Lie algebra $\mathfrak{g}$. Show by explicit calculation that a non-abelian gauge field configuration of the form
$$A_\mu = i h (\partial_\mu h^{-1}) \ ,$$
where $h(x)$ is a (space-time dependent) element of G, has vanishing field strength:
$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = 0 \ .$$
Can you think of a simpler argument to reach the same conclusion?

Ex 32 Show that, for any irreducible representation $\mathbf{r}$ of the gauge group G, the Yang-Mills Lagrangian can be written as
$$\mathcal{L}_{YM} = -\frac{1}{2g^2_{YM}} \frac{1}{T(\mathbf{r})} \mathrm{tr}_{\mathbf{r}}(F^{(\mathbf{r})}_{\mu\nu} F^{(\mathbf{r})\mu\nu}) \ ,$$
where the Dynkin index
$$T(\mathbf{r}) := \frac{C(\mathbf{r})}{C(\mathrm{fund})}$$
of the irreducible representation $\mathbf{r}$ is invariant under changes of normalization of the Lie algebra.

Ex 33 1. Express the Lagrangian density

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2g_{YM}^2}\,\text{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{\theta}{16\pi^2}\,\text{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$$

in terms of A_μ^a and the structure constants $f_{ab}{}^c$, and identify quadratic terms in the gauge field (and derivatives thereof), and cubic and quartic terms, which represent interactions.

2. Show that the theta term action

$$S_\theta[A] = \frac{\theta}{16\pi^2}\int d^4x\ \text{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) = \frac{\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4x\ \text{tr}(F_{\mu\nu}F_{\rho\sigma})$$

can be written as a surface (or 'boundary') term:

$$S_\theta[A] = \frac{\theta}{8\pi^2}\int d^4x\ \partial_\mu K^\mu\ ,$$
$$K^\mu = \epsilon^{\mu\nu\rho\sigma}\,\text{tr}(A_\nu\partial_\rho A_\sigma - \frac{2i}{3}A_\nu A_\rho A_\sigma)\ .$$

3. Show that the equations of motion obtained from the Lagrangian density $\mathcal{L}_{\text{gauge}}$ are

$$D_\mu F^{\mu\nu} \equiv \partial_\mu F^{\mu\nu} - i[A_\mu, F^{\mu\nu}] = 0\ .$$

4. Show, without using the equations of motion, that the Bianchi identity

$$D_\mu \tilde{F}^{\mu\nu} = 0\ .$$

holds.

Ex 34 Consider a gauge theory with Lagrangian density

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} \\ &= -\frac{1}{2g_{YM}^2}\,\text{tr}(F_{\mu\nu}F^{\mu\nu}) + \frac{\theta}{16\pi^2}\,\text{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) - (D_\mu\phi)^\dagger D^\mu\phi - V(\phi,\phi^\dagger)\ ,\end{aligned}$$

where the scalar potential $V(\phi,\phi^\dagger)$ is a gauge invariant function of ϕ, which transforms in the fundamental representation of the gauge group G, and of $\phi^\dagger = \overline{\phi}^T$.

1. Show that the equations of motion are

$$D_\mu D^\mu\phi = \frac{\partial V}{\partial\phi^\dagger}$$
$$D_\nu F^{\mu\nu} = g_{YM}^2 J^\mu$$

for a current $J_\mu = J_\mu^a t_a$ that you should find.

2. Show that the current J^μ transforms as

$$J^\mu \mapsto g J^\mu g^{-1}$$

under a gauge transformation with group element $g = g(x)$, and that it is covariantly conserved, namely

$$D_\mu J^\mu = 0 \ .$$

Ex 35 Let the gauge group G be one of the classical compact simple Lie groups in exercise 26. For each of the three infinite families $G = SU(N)$, $G = SO(N)$ and $G = USp(2N)$:

1. Write down the finite and the infinitesimal gauge transformations for the gauge field A_μ, the field strength $F_{\mu\nu}$, a scalar field χ transforming in the fundamental representation, and a scalar field Φ transforming in the adjoint representation.
2. Write down a gauge invariant Lagrangian density for the above fields, including kinetic terms and mass terms for the scalar fields. You can ignore the theta term and assume the reality property $\chi = \bar{\chi}$ for $G = SO(N)$, and $\chi = J\bar{\chi}$ for $G = USp(2N)$.

8 Topological solitons in gauge theories

Ex 36 The Georgi-Glashow model, or $SU(2)$ adjoint Higgs model, consists of a scalar field $\Phi = \phi^a \sigma_a$ transforming in the adjoint representation of an $SU(2)$ gauge group, with Lagrangian density

$$\mathcal{L} = -\frac{1}{2g_{YM}^2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \mathrm{tr}((D_\mu\Phi)(D^\mu\Phi)) - V(\Phi) \ ,$$

where the gauge invariant scalar potential is

$$V(\Phi) = \lambda\left(\frac{1}{2}\,\mathrm{tr}(\Phi^2) - v^2\right)^2$$

and

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ D_\mu\Phi &= \partial_\mu\Phi - i[A_\mu, \Phi] \ . \end{aligned}$$

1. Show that the energy of static field configurations (or 'static energy') in the gauge $A_0 = 0$ is

$$E = \int d^3x \left[\frac{1}{g_{YM}^2}\,\mathrm{tr}(B_iB_i) + \mathrm{tr}((D_i\Phi)(D_i\Phi)) + V(\Phi)\right] \ .$$

2. Show that the static energy is minimized by constant field configurations with $\boldsymbol{A} = 0$ and

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = v^2 \ ,$$

up to a gauge transformation.

3. Show that by a constant gauge transformation, any field configuration of least energy (or vacuum) as in part 2 can be written in the form

$$\Phi = v\sigma_3 = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} \ ,$$

and that this is invariant under a $H = U(1)$ subgroup of the gauge group $G = SU(2)$. (One says that the gauge symmetry G is spontaneously broken to H.)

4. * Expand the fields about this vacuum configuration (namely set $X = X_{vacuum} + \delta X$) and substitute this expansion in the Lagrangian density, assuming that there are no gauge fields so that the symmetry G is a global symmetry. Show that the components of Φ in the directions of the broken symmetries (*e.g.* ϕ^1 and ϕ^2 in the parametrization of part 2) are massless, having no quadratic scalar potential terms. (These massless scalar fields are called Nambu-Goldstone bosons in physics. They arise whenever a continuous global internal symmetry is spontaneously broken.)

5. * Repeat the exercise for the full Georgi-Glashow model, which has gauge field and $G = SU(2)$ gauge symmetry. Show that the would-be Nambu-Goldstone bosons can be eliminated by a gauge transformation, and that the gauge fields for the spontaneously broken part of the gauge group gain a mass. (This is the key idea of the Higgs mechanism.)

Ex 37 Consider again the Georgi-Glashow model of exercise 36, with $\Phi = \phi^a \sigma_a$, where (σ_a) are the Pauli matrices. Let

$$\Phi_\infty(\theta, \varphi) := \lim_{r \to \infty} \Phi(r, \theta, \varphi)$$

be the limit of Φ at spatial infinity. The boundary conditions require that

$$\frac{1}{2} \operatorname{tr}(\Phi_\infty^2) = (\phi_\infty^1)^2 + (\phi_\infty^2)^2 + (\phi_\infty^3)^2 = v^2 \ ,$$

hence Φ_∞ is a map from an S^2 of unit radius to an S^2 of radius v.

1. Show that the Georgi-Glashow model has a current

$$j^\mu = \epsilon^{\mu\nu\rho\sigma} \epsilon_{abc} (\partial_\nu \phi^a)(\partial_\rho \phi^b)(\partial_\sigma \phi^c)$$

which is conserved irrespective of the equations of motion. Calculate the associated conserved charge Q.

2. Show that $Q = c\nu$ where ν is the topological degree

$$\nu := \frac{1}{4\pi v^3} \int_{S^2_\infty} \frac{1}{2} \epsilon^{ijk} \epsilon_{abc} \phi^a_\infty \partial_j \phi^b_\infty \partial_k \phi^c_\infty \ d^2\sigma_i$$

of the map $\Phi_\infty = \phi^a_\infty \sigma_a$, for a suitable constant c that you should find.

3. Calculate the degree of the map

$$(\phi^1_\infty, \phi^2_\infty, \phi^3_\infty) = v(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \ .$$

4. Define

$$F^{U(1)}_{\mu\nu} := \frac{1}{2v} \operatorname{tr}(\Phi_\infty F_{\mu\nu})$$

to be the field strength of the unbroken $H = U(1)$ subgroup of the gauge group $G = SU(2)$. Show that the magnetic charge

$$m^{U(1)} := \frac{1}{2\pi} \int_{S^2_\infty} \vec{B}^{U(1)} \cdot d^2\vec{\sigma}$$

of the unbroken $U(1)$ is proportional to the topological degree ν of Φ_∞, and find the proportionality factor.

9 Mock exam questions

Q1 (10 marks)
A $U(1)$ gauge theory in two space and one time dimensions has Lagrangian density

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho - A_\mu J^\mu \ ,$$

where $\epsilon^{\mu\nu\rho}$ is the completely antisymmetric tensor in three indices with normalization $\epsilon^{012} = 1$, g and κ are constants, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and J^μ is an external (*i.e.* non-dynamical) current that the gauge field A_μ is coupled to.

1. Write down the Euler-Lagrange equation for the gauge field A_μ and show that it implies that the current J^μ is conserved.
2. Show that the variation of the Lagrangian density under a $U(1)$ gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu \alpha$ is a total derivative term, that you should find.

Q2 (10 marks)
The gauge covariant derivative in the fundamental representation of a gauge group G, D_μ, transforms as $D_\mu \mapsto U D_\mu U^{-1}$ under a gauge transformation by the group element $U \in G$.

1. How do a field ψ in the fundamental representation and a field ϕ in the adjoint representation of G transform under a gauge transformation?
2. The gauge covariant derivative in the adjoint representation $D_\mu^{(\mathrm{ad})}$ is defined by its action
$$D_\mu^{(\mathrm{ad})}\phi := [D_\mu, \phi]$$
on any field ϕ in the adjoint representation. Find how $D_\mu^{(\mathrm{ad})}$ transforms under a gauge transformation.
3. Calculate $[D_\mu^{(\mathrm{ad})}, D_\nu^{(\mathrm{ad})}]\phi$ and express the result in terms of known quantities. You may use without proof that the field strength acting in the fundamental representation is $F_{\mu\nu} = i[D_\mu, D_\nu]$.

Q3 (15 marks)
A circle of circumference L can be parametrized by a real coordinate x with the periodic identification $x \sim x + L$. Well-defined (or 'single-valued') functions on the circle are periodic functions: $f(x+L) = f(x)$. One can generalize this to a two-dimensional torus T^2, which is the product of two circles, with coordinates $x^\mu = (x, y)$ and identifications

$$(x, y) \sim (x + L_x, y) \sim (x, y + L_y) \ .$$

Consider a $U(1)$ gauge field (or vector potential) on T^2, with constant magnetic field

$$F_{xy} = \partial_x A_y - \partial_y A_x = B_0 = \text{const} \ .$$

1. Find the general solution for a $U(1)$ gauge field A_μ corresponding to this constant magnetic field. Then specialize to the case that $A_x = 0$. Finally, specialize to $A_x = 0$ in the Lorenz gauge $\partial_i A_i = 0$. You may use this final solution in part 2.
2. Impose that that the gauge field A_μ be periodic along the two circles of the torus, up to a $U(1)$ gauge transformation with single-valued gauge parameter $g = e^{i\alpha}$. Show that this implies that the constant magnetic field B_0 is a multiple of a fundamental unit, that you should find.
3. Now assume that $B_0 = 0$, and let α_x, α_y be constants. Find A_μ such that

$$\int_0^{L_x} dx \ A_x = \alpha_x \ , \quad \int_0^{L_y} dy \ A_y = \alpha_y \ .$$

Q4 (15 marks)
Consider three-dimensional space $\mathbb{R}^3$, with Euclidean coordinates (x, y, z) and with polar coordinates (r, θ, φ). Here $r \in \mathbb{R}_+$ is a radial coordinate and (θ, φ), with $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi]$, are angular coordinates which parameterize a unit 2-sphere S^2. To cover S^2 we use a northern patch U_+, which excludes the south pole, and a southern patch U_-, which excludes the north pole. Consider the gauge field configuration defined by the vector potentials

$$A_x^+ = -\frac{m}{2}\frac{y}{r(r+z)} \ , \quad A_y^+ = \frac{m}{2}\frac{x}{r(r+z)} \ , \quad A_z^+ = 0$$

in the northern patch $\mathbb{R}_+ \times U_+$, and

$$A_x^- = \frac{m}{2}\frac{y}{r(r-z)} \ , \quad A_y^- = -\frac{m}{2}\frac{x}{r(r-z)} \ , \quad A_z^- = 0$$

in the southern patch $\mathbb{R}_+ \times U_-$.

1. Switching from cartesian coordinates to polar coordinates, and using

$$A^\pm = A_x^\pm dx + A_y^\pm dy + A_z^\pm dz = A_r^\pm dr + A_\theta^\pm d\theta + A_\varphi^\pm d\varphi \ ,$$

find expressions for $A^\pm$ in polar coordinates, and calculate $A^+ - A^-$ on the overlap of the two patches.

2. Calculate the magnetic flux through a 2-sphere of radius R centred at the origin,

$$\int_{S^2_R} \vec{B} \cdot d\vec{\sigma} = \int_{S^2_R} F_{\theta\varphi} d\theta d\varphi \ ,$$

where $\vec{B} = \vec{B}^{\pm} := \nabla \times \vec{A}^{\pm}$ in the regions where the two vector potentials $\vec{A}^{\pm}$ are defined, in both of the following two ways:

(a) By finding the magnetic field and integrating it over the 2-sphere;

(b) By reducing the surface integral over the 2-sphere to a line integral over its equator.

How does the flux depend on the radius R?

3. Show that the energy of this gauge field configuration

$$E = \frac{1}{2g^2} \int_{\mathbb{R}^3} d^3x \ \vec{B}^2 = \frac{1}{4g^2} \int_{\mathbb{R}^3} d^3x \ F_{ij} F^{ij}$$

is infinite. You may use without proof that $F^{\theta\varphi} = r^{-4} (\sin\theta)^{-2} F_{\theta\varphi}$.

Q5 (15 marks)
Scalar chromodynamics is a gauge theory with gauge group $G = SU(3)$ and N_f 'flavours' of scalar fields ϕ_i ($i = 1, \ldots, N_f$) transforming in the fundamental representation of the gauge group.

1. Write down a gauge invariant action including kinetic terms for the $SU(3)$ gauge field A_μ and the N_f scalars ϕ_i but no scalar potential, and check explicitly that this action is invariant under $SU(3)$ gauge transformations.
2. What is the global symmetry of the gauge theory with action written in part 1?
3. Write down the most general gauge invariant real scalar potential $V(\phi, \phi^\dagger)$ of degree at most 3 in ϕ and $\phi^\dagger$.
4. Repeat the exercise in part 3 under the further assumption that the global symmetry found in part 2 is preserved.