Geometry of Mathematical Physics III
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Chapter 5

$U(1)$ gauge theory reloaded

We will start the term with a reminder of the formulation of $U(1)$ gauge theories. I will introduce my conventions, which slightly differ from Andreas' conventions, and I will make a number of remarks, some of which you might be familiar with already, and some of which you won't. This will set the stage for the formulation of non-abelian gauge theories in the chapter 6, and along the way I will also give a few appetizers of what's to come later in the term.

5.1 $U(1)$ global symmetry

Consider a complex scalar field $\phi(x)$.[1] The action[2]

$$\begin{aligned} S_0[\phi, \bar{\phi}] &= \int d^4x \; \mathcal{L}_0(\phi, \bar{\phi}, \partial_\mu \phi, \partial_\mu \bar{\phi}) \;, \\ \mathcal{L}_0 &= -|\partial_\mu \phi|^2 - V(\phi, \bar{\phi}) = -|\partial_\mu \phi|^2 - U(|\phi|^2) \\ &= |\dot{\phi}|^2 - |\nabla \phi|^2 - U(|\phi|^2) \end{aligned} \tag{5.1}$$

is invariant under **global** $G = U(1)$ transformations

$$g: \quad \phi(x) \mapsto e^{i\alpha} \phi(x)$$

[1]Recall that mathematically, this is a map from Minkowski space-time $\mathbb{R}^{1,3}$ to $\mathbb{C}$, which associates a complex number to each point in space-time:

$$\begin{array}{cccc} \phi\colon & \mathbb{R}^{1,3} & \to & \mathbb{C} \\ & x^\mu & \mapsto & \phi(x) \end{array}$$

Greek indices $\mu, \nu, \ldots$ are space-time indices running from 0 to 3. (Roman indices $i, j, \ldots$ are spatial indices running from 1 to 3. Index 0 is for time.)

Unless we explicitly state otherwise, we will typically assume that all fields are smooth.

[2]$|\partial_\mu \phi|^2$ is a short-hand notation for $\partial_\mu \bar{\phi} \partial^\mu \phi$, where Einstein summation convention (repeated indices are summed over) is understood. Recalling that we work with Minkowski metric $[\eta_{\mu\nu}] = (-1, +1, +1, +1)$, this means that $|\partial_\mu \phi|^2 = -|\partial_0 \phi|^2 + |\partial_i \phi|^2 = -|\dot{\phi}|^2 + |\nabla \phi|^2$.

where $\alpha \sim \alpha + 2\pi$ is a **constant** parameter, and $g = e^{i\alpha} \in U(1)$ is a constant group element. The requirement of $U(1)$ invariance restricts the scalar potential $V(\phi, \bar{\phi})$ to only depend on the invariant $|\phi|^2$. Because the scalar field ϕ is multiplied by a single power of the $U(1)$ group element $g = e^{i\alpha}$, we say that it has charge 1.

REMARKS:

1. The continuous $U(1)$ symmetry ensures the existence of a **conserved current**

$$\begin{aligned} j^\mu &= i(\bar{\phi}\partial^\mu\phi - \phi\partial^\mu\bar{\phi}) \\ \partial_\mu j^\mu &= 0 \end{aligned} \tag{5.2}$$

   and of a **conserved charge**

$$\begin{aligned} Q &= \int d^3x \; j_0 \\ \frac{d}{dt}Q &= 0 \end{aligned} \tag{5.3}$$

   by Noether's theorem.

2. A **global symmetry** relates **physically distinct configurations**.

*** EXERCISE**:

[Ex 1] Consider a field theory with action (5.1) and scalar potential

$$V(\phi, \bar{\phi}) = \lambda(|\phi|^2 - a^2)^2 \; ,$$

with parameters $\lambda, a > 0$, see figure 5.1.

1. Calculate the energy (or "Hamiltonian")

$$\begin{aligned} E &= \int d^3x \; \left(|\partial_0\phi|^2 + |\partial_i\phi|^2 + V(\phi, \bar{\phi})\right) \\ &= \int d^3x \; \left(|\dot{\phi}|^2 + |\nabla\phi|^2 + V(\phi, \bar{\phi})\right) \; . \end{aligned}$$

   You may use the relation between the Lagrangian and Hamiltonian densities, or calculate the Noether charge associated to invariance under time translations $t \mapsto t + c$.

2. Show that the configurations of least energy ("vacua", or "ground states") parametrize a circle in field space.

3. Show that different vacua are related by global $U(1)$ transformations.

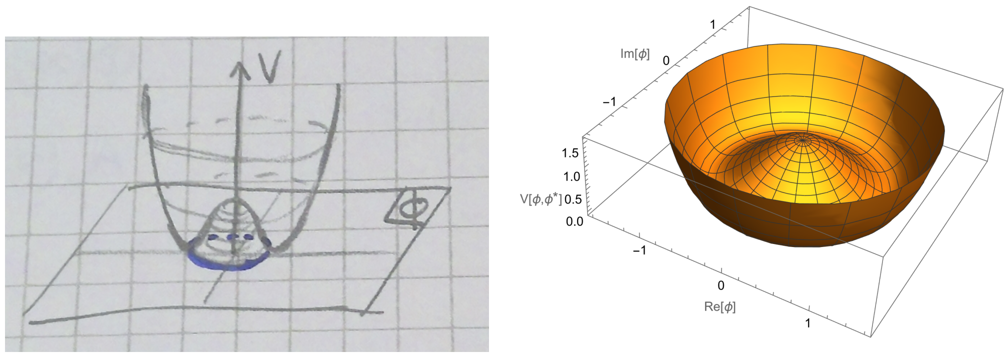

Figure 5.1: The scalar potential $V(\phi, \bar{\phi}) = \lambda(|\phi|^2 - a^2)^2$.

5.2 $U(1)$ gauge symmetry

To make the global symmetry local, or a gauge symmetry, we promote the constant parameter α to a function of spacetime $\alpha(x)$. For subtle reasons that we might return to later, the parameter $\alpha(x)$ of a gauge transformation should approach 0 (sufficiently fast) at infinity.

The action

$$\begin{aligned} S[\phi, \bar{\phi}, A_\mu] &= \int d^4x\, \mathcal{L}(\phi, \bar{\phi}, A_\nu, \partial_\mu \phi, \partial_\mu \bar{\phi}, \partial_\mu A_\nu)\ , \\ \mathcal{L} &= \mathcal{L}_0(\phi, \bar{\phi}, D_\mu \phi, \overline{D_\mu \phi}) + \mathcal{L}_{\text{Maxwell}}(\partial_\mu A_\nu) \\ &= -\overline{D_\mu \phi} D^\mu \phi - U(|\phi|^2) - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu}\ , \end{aligned} \tag{5.4}$$

where A_μ is a real **gauge field** (or mathematically, a “gauge connection”) and

$$\begin{aligned} D_\mu \phi &:= (\partial_\mu - iA_\mu)\phi && \textbf{covariant derivative} \text{ of } \phi \\ F_{\mu\nu} &:= \partial_\mu A_\nu - \partial_\nu A_\mu && \textbf{field strength} \text{ of } A_\mu \end{aligned} \tag{5.5}$$

is invariant under $G = U(1)$ **gauge** transformations

$$\begin{aligned} \phi(x) &\mapsto e^{i\alpha(x)} \phi(x) \\ A_\mu(x) &\mapsto A_\mu(x) + \partial_\mu \alpha(x)\ . \end{aligned} \tag{5.6}$$

REMARKS:

1. To linear order in the gauge field A_μ

$$\mathcal{L} = \mathcal{L}_0 - j^\mu A_\mu + \dots \tag{5.7}$$

The scalar field is coupled (via covariant derivatives) to the gauge field A_μ, and not to the field strength $F_{\mu\nu}$. To leading order, the gauge field A_μ couples directly to the conserved current j^μ of the theory with $U(1)$ global symmetry, which is built out of the scalar field. This type of coupling is called the **minimal coupling**.

A common alternative normalization to the one we use is obtained by rescaling the gauge field by one power of the gauge coupling: $A_\mu \to gA_\mu$. In that normalization the Lagrangian density is

$$\begin{aligned}\mathcal{L} &= -\left((\partial^\mu + igA^\mu)\bar{\phi}\right)(\partial_\mu - igA_\mu)\phi - U(|\phi|^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \mathcal{L}_0 - gj^\mu A_\mu + \ldots\end{aligned}$$

where the ellipses denote terms quadratic in the gauge field. This normalization makes it clear that the **gauge coupling** g controls the strength of the coupling between the conserved current j^μ of the theory with $U(1)$ global symmetry and the gauge field A_μ. In the following we will typically stick to the convention in which the gauge coupling g appears in front of the kinetic term for the gauge field, rather than inside gauge covariant derivatives.

2. The group of gauge transformations

$$\mathcal{G} = \mathcal{U}(1) := \left\{ \begin{array}{cccc} g: & \mathbb{R}^{1,3} & \to & G = U(1) \\ & x^\mu & \mapsto & g(x) = e^{i\alpha(x)} \end{array} \right\} \tag{5.8}$$

is infinite-dimensional, since it associates independent transformations $g(x)$ for the fields at different points x^μ, and there are infinitely many points in space-time. We use calligraphic letters to distinguish the gauge group from the associated finite-dimensional (for $G = U(1)$, one-dimensional) Lie group. Later on, once we have familiarized ourselves with this distinction, we might drop this notation and simply use G for the gauge group, with a common abuse of notation.

3. A **“gauge symmetry”** relates **physically equivalent configurations**, which are to be **identified**. The term “ gauge symmetry” is therefore a misnomer: it is **not a symmetry**, but rather a **redundancy** in our description of the theory.

The identification of field configurations which differ by a gauge transformation[3] leads to non-trivial topological properties of gauge fields, which in turn ensure the existence of topological solitons and instantons, non-trivial gauge field configurations which are stable for topological reasons. We will study these configurations in chapter 9.

From now on we omit writing the dependence on the space-time coordinate x. It is understood that all fields and all gauge transformation parameters depend on x.

[3] See section 2.6 of [Manton and Sutcliffe, 2004] if you want to read more about this.

4. Under a $U(1)$ gauge transformation (5.6),

$$\begin{aligned} D_\mu\phi &\mapsto e^{i\alpha} D_\mu \phi \ , \\ F_{\mu\nu} &\mapsto F_{\mu\nu} \end{aligned} \tag{5.9}$$

We say that the covariant derivative $D_\mu\phi$ of ϕ is **gauge covariant**, because it transforms in a representation of G for all x (the same representation of ϕ, namely the charge 1 representation here), and that the field strength $F_{\mu\nu}$ is **gauge invariant**, because it does not change under a gauge transformation (in fancy language, it transforms in the trivial, or "singlet", representation).

5. It proves very useful to think of the covariant derivative $D_\mu = \partial_\mu - iA_\mu$ as a **differential operator**, which acts on everything to its right. The partial derivative ∂_μ acts by differentiating all that appears to its right, which the gauge field A_μ, like all functions of x, acts by multiplying all that appears to its right. Requiring that under a $U(1)$ gauge transformation

$$D_\mu \equiv \partial_\mu - iA_\mu \mapsto D'_\mu \equiv \partial_\mu - iA'_\mu = e^{i\alpha} D_\mu e^{-i\alpha} \ , \tag{5.10}$$

so that

$$D_\mu\phi \mapsto e^{i\alpha} D_\mu e^{-i\alpha} e^{i\alpha}\phi = e^{i\alpha} D_\mu \phi \tag{5.11}$$

as desired, implies the gauge transformation of the gauge field

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu \alpha \tag{5.12}$$

and vice versa.

Proof. For the implication (5.10) $\Rightarrow$ (5.12), we expand (5.10) and act with ∂_μ on everything to its right. There are two options: either ∂_μ acts on $e^{-i\alpha}$, which produces the function $(\partial_\mu e^{-i\alpha}) = -i\, e^{-i\alpha}(\partial_\mu\alpha)$, or ∂_μ goes through $e^{-i\alpha}$, which produces the differential operator $e^{-i\alpha}\partial_\mu$.[4] Then we find

$$\begin{aligned} D_\mu \equiv \partial_\mu - iA_\mu \mapsto D'_\mu \equiv \partial_\mu - iA'_\mu &= e^{i\alpha}(\partial_\mu - iA_\mu)e^{-i\alpha} \\ &= e^{i\alpha}e^{-i\alpha}(\partial_\mu\alpha) + e^{i\alpha}e^{-i\alpha}\partial_\mu - ie^{i\alpha}e^{-i\alpha}A_\mu \\ &= \partial_\mu - i(A_\mu + \partial_\mu\alpha) \ , \end{aligned}$$

which comparing the initial expression and the final results implies

$$A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu\alpha \ .$$

[4]If you are confused by these statements and manipulations, act with the differential operator on any smooth test function $f(x)$. If X and Y are two differential operators, then $X = Y$ iff $Xf = Yf$ for all smooth test functions. Similarly $X \mapsto Y$ iff $Xf \mapsto Yf$ for all smooth test functions.

I leave it to you to prove the opposite implication (5.10) $\Leftarrow$ (5.12). That is simply a reminder from last term. You can assume that D_μ acts on a field of charge 1.

Furthermore, defining the commutator $[X, Y] := XY - YX$, we have

$$[D_\mu, D_\nu] = -iF_{\mu\nu} \ , \tag{5.13}$$

so the field strength controls the non-commutativity of covariant derivatives. We will learn more about this later when we study "curvatures", which is the mathematical term for the mathematical object that the field strength $F_{\mu\nu}$ is.

Proof. **[Ex 2]**

6. The gauge field A_μ is only defined **locally**, namely **in a patch**, which we take to be contractible to a point so that the Poincaré lemma applies. Indeed, in the gauge theory formulation of electromagnetism, the Bianchi identity $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$ implies $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ only if the Poincaré lemma applies.

   What this means is the following. Consider two patches $U^{(1)}$ and $U^{(2)}$ with a non-trivial overlap $U^{(1)} \cap U^{(2)} \neq \varnothing$. Then the gauge fields $A^{(1)}_\mu$ and $A^{(2)}_\mu$ defined in the two patches are related by a gauge transformation

$$A^{(1)}_\mu = A^{(2)}_\mu + \partial_\mu \alpha^{(12)}$$

   on the overlap $U^{(1)} \cap U^{(2)}$.[5] Mathematically, the gauge transformation parameter $\alpha^{(12)}$ that relates the gauge fields in the two patches is called a "transition function". We will learn more about how gauge fields/connections defined in different patches are glued together as we move to a different patch in chapter 7. Charged fields are also defined locally, in patches. For consistency, they also transform by a gauge transformation when we switch to another patch. We will study this in chapter 8.

   This local definition of A_μ is responsible for most of the topological and geometric properties of gauge theories. To give you an appetizer, consider a space-time of the form $\mathbb{R} \times (\mathbb{R}^3 \backslash p)$, where the first factor of $\mathbb{R}$ is parametrized by time, and the second factor is space, which is flat Euclidean space $\mathbb{R}^3$ except that we excise the point p (we could equally excise a 3-ball).[6] It turns out that this space-time is not contractible to a point, but only to a 2-sphere surrounding the point p. (Perhaps you can figure it in your mind.

[5] Naively you might want to impose the simpler identification $A^{(1)}_\mu = A^{(2)}_\mu$, but taking into account that gauge fields are only defined modulo gauge transformations, one is led to the more general (and mathematically correct) identification in the main text. It took physicists several decades to appreciate this point.

[6] I use the symbol $\backslash$ to denote set difference. If you are use to the ordinary sign to denote set difference as well, please let me know and I'll change my notation accordingly.

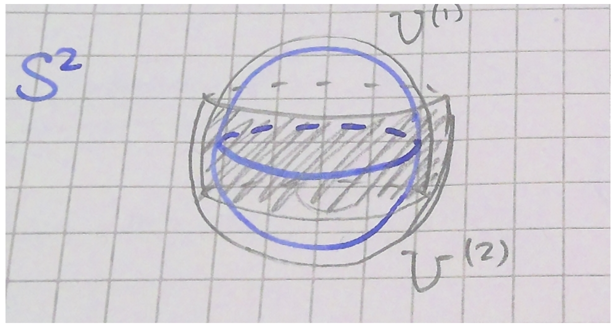

Figure 5.2: Two patches which cover a 2-sphere S^2, and their overlap.

If not, just trust me for now.) Last term, when you learned about stereographic projections, you saw that a 2-sphere can be covered by two patches, see figure 6. For instance, we can take patch $U^{(1)}$ to cover everything north of the southern tropic, and patch $U^{(2)}$ to cover everything south of the northern tropic. The two patches overlap in the region between the two tropics near the equator, so we need to specify how the gauge field in the northern patch and the gauge field in the southern patch are related in this region where both are defined. As we will see, this freedom allows us to define a magnetic monopole, namely a pointlike magnetic charge, sitting at point p. This is very surprising, because Maxwell's equations allow electric charge densities but not magnetic charge densities in the right-hand sides. As we will study in chapter 9, we can by-pass this limitation by exploiting the topology of the gauge field.

$*$ EXERCISE:

[Ex 3] So far I have assumed for simplicity that the complex scalar field ϕ has charge 1. Go through this chapter and work out how all formulae change if ϕ has charge q rather than charge 1.

5.3 Gauge redundancy and gauge fixing

A good reference for this topic is section 6 of David Tong's QFT lecture notes [Tong, 2006].

Let us start from the equations of motion (EoM) of the theory of **scalar electrodynamics**, which is described by the action (5.4). We recall here the Lagrangian density

$$\mathcal{L} = -|D_\mu \phi|^2 - V(\bar{\phi}, \phi) - \frac{1}{4g^2} F_{\mu\nu}^2 \, ,$$

where $F_{\mu\nu}^2 \equiv F_{\mu\nu} F^{\mu\nu}$ *etc*, and the scalar potential takes the form $V(\bar{\phi}, \phi) = U(|\phi|^2)$ to ensure gauge invariance. Then the Euler-Lagrange equations are

$$\begin{aligned} &1) \quad D_\mu D^\mu \phi = -\frac{\partial V}{\partial \bar{\phi}} \equiv -U'(|\phi|^2)\phi \\ &2) \quad\ \ \partial_\mu F^{\mu\nu} = g^2 J^\nu \end{aligned} \tag{5.14}$$

where

$$J_\mu = i(\bar{\phi} D_\mu \phi - \phi D_\mu \bar{\phi}) = j_\mu + 2A_\mu |\phi|^2 \tag{5.15}$$

is a conserved current. The EoM for $\bar{\phi}$ is the complex conjugate of the EoM for ϕ, so I will not write it explicitly. Note that upon gauging the global U(1) symmetry, the conserved current j_μ (5.3) of the scalar field theory with global $U(1)$ symmetry gets a correction term due to the presence of the gauge field A_μ in the covariant derivatives.

Proof. **[Ex 5]**

Let us now consider the transformation properties of the EoM (5.1) under a $U(1)$ gauge transformation (5.6). The equations transform as

$$\begin{aligned} &1) \mapsto e^{i\alpha} 1) \qquad \text{(gauge covariant)} \\ &2) \mapsto 2) \qquad\quad \text{(gauge invariant)} \end{aligned} \tag{5.16}$$

Therefore, if a field configuration (ϕ, A_μ) solves the EoM (5.14), then any gauge transformed field configuration $(\phi' = e^{i\alpha}\phi, A'_\mu = A_\mu + \partial_\mu \alpha)$ also solves the EoM (5.14): the EoM only determine EoM (ϕ, A_μ) up to a gauge transformation.

Given some initial data $(\phi^{(0)}, A^{(0)}_\mu$ specifying the field configuration at an initial time t_0, we cannot uniquely determines the field configuration (ϕ, A_μ) at a later time $t > t_0$. Indeed $(\phi' = e^{i\alpha}\phi, A'_\mu = A_\mu + \partial_\mu \alpha)$ is as good a solution of the EoM as (ϕ, A_μ), and obeys the same initial condition provided that the gauge parameter $\alpha = 0$ at the initial time t_0.

We appear to be in trouble: we would like the EoM to define a **well-posed initial value problem** and determine **uniquely** physically observable fields at later times. This is not the case if we regard field configurations which differ by a gauge transformation as physically inequivalent. If instead we declare field configurations which differ by a gauge transformation to be physically equivalent, then the issue disappears and the initial value problem is well-posed. We will therefore **identify field configurations related by a gauge transformation**,

$$(\phi, A_\mu) \ \sim \ (\phi' = e^{i\alpha}\phi, A'_\mu = A_\mu + \partial_\mu \alpha) \ . \tag{5.17}$$

Physically observable quantities must then be **gauge invariant**, such as for example the field strength $F_{\mu\nu}$, the magnitude of the scalar field $|\phi|^2$, or the conserved current J_μ. This explains remark 3 in the previous section.

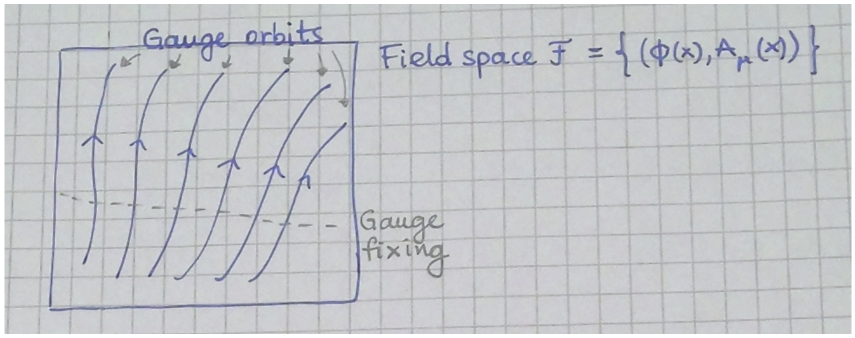

Figure 5.3: The space of all field configurations decomposes into the disjoint union of gauge orbits, each represents a single physical configuration. A complete gauge fixing selects a single representative for each orbit.

The picture to keep in mind for gauge theories is that field space $\mathcal{F} = \{\phi(x), A_\mu(x)\}$ is foliated[7] by **gauge orbits** traced by the action of the gauge group

$$\mathcal{G} \cdot (\phi(x), A_\mu(x)) = \{(e^{i\alpha(x)}\phi(x), A_\mu(x) + \partial_\mu \alpha(x) \mid \alpha(x) \sim \alpha(x) + 2\pi\} \, .$$

In down to earth terms, a gauge orbit simply consists of all the field configurations which are related by a gauge transformation.

Then the identification (5.17) of field configurations related by gauge transformations states the correspondence[8]

$$\text{Physical configuration} \quad \longleftrightarrow \quad \text{Gauge orbit} \, .$$

Rather than working with the redundant description of field space $\mathcal{F}$ subject to the gauge symmetry $\mathcal{G}$, it is often useful to **"fix a gauge"** (or pick a gauge, that is, picking a single **representative** for each gauge orbit. Any representative does the job – after all any two representatives of a given gauge orbit are physically equivalent – but we need to ensure that the gauge fixing cuts each orbit once and only once, as in figure 5.3. If that is not the case, and there is some leftover gauge symmetry that is not fixed, we refer to the gauge fixing as partial or incomplete, and further conditions must be specified in order to have a complete gauge fixing. The topic of gauge fixing is rather technical, and plays an important role in the quantization of gauge theories. Here we will content ourselves with giving a few standard examples of (partial) gauge fixing, which may be useful later on.

[7]Foliation is a mathematical term, from 'folia', Latin for 'leaf'. You can look up the technical definition if you are interested. For our purposes, you can take it to mean that field space is a union of disjoint orbits of the gauge group.

[8]If you are formally minded, you would say that the physical configuration space $\mathcal{C}$ is the quotient of the field space $\mathcal{F}$ by the gauge group $\mathcal{G}$,

$$\mathcal{C} = \mathcal{F}/\mathcal{G} \, ,$$

namely the set of equivalence classes of field configurations under the equivalence relation (5.17).

EXAMPLES:

1. **Lorenz gauge**:
   This gauge is defined by imposing the constraint

$$\partial_\mu A^\mu = 0 \tag{5.18}$$

   on the gauge field 4-vector A_μ. This can always be achieved. Indeed, if we are given a representative A_μ which does not obey the Lorenz gauge condition (5.18), then we can find another representative $A'_\mu = A_\mu + \partial_\mu \alpha$ in the same gauge orbit which obeys the Lorenz gauge constraint

$$0 = \partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \alpha \tag{5.19}$$

   by picking α to be a solution of the

$$\partial_\mu \partial^\mu \alpha = -\partial_\mu A^\mu \ , \tag{5.20}$$

   which exists.[9]

   Let us discuss pros and cons of the Lorenz gauge. The main advantage of the Lorenz gauge is that the constrain (5.18) is **Lorentz invariant**.[10] The main disadvantage of the Lorenz gauge is that it only fixes the gauge partially. Indeed, if we are in Lorenz gauge we are free to perform gauge transformations with parameters α such that $\partial_\mu \partial^\mu \alpha = 0$ and we will remain in the Lorenz gauge. (This corresponds to adding a solution of the homogeneous equation in .)

2. **Coulomb gauge** (or **radiation gauge**):
   This gauge is defined by imposing the constraint

$$\nabla \cdot \vec{A} = 0 \tag{5.21}$$

   on the vector potential $\vec{A}$, which is the spatial part of the 4-vector A_μ. This can always be achieved, by a similar reasoning to above.

   Compared to the Lorenz gauge, the Coulomb gauge has the clear drawback of not being Lorentz covariant. So this gauge fixing spoils the manifest relativistic symmetry of the formalism, which is not ideal. (The physics of the system remains Lorentz invariant,

[9] Here the right-hand side $-\partial_\mu A^\mu$ is given and acts as a source in a relativistic Poisson equation for α. Solutions can be found by the method of Green's functions.

[10] The Lorenz gauge is due to the Danish physicist Ludvig Lorenz, not to be confused with the more famous Dutch physicist Hendrik Lorentz, who is responsible for the Lorentz transformations which leave the laws of special relativity invariant, as well as for introducing the Lorentz force which acts on relativistic particles moving in a magnetic field. Click on the names of the physicists to see who is who.

because gauge transformations are unphysical.) Another drawback, in common with the Lorenz gauge, is that the Coulomb gauge constraint (5.21) only fix the gauge partially. The argument is the same as for the Lorenz gauge, except that we are using spatial indices only instead of full space-time indices.

On the other hand, a pro of the Coulomb gauge is that the temporal component A_0 of the gauge potential (aka the 'electric scalar potential' in electromagnetism) is determined by the charge density $\rho = J^0$ as in electrostatics:

$$A_0(t,\vec{x}) \propto \int d^3x' \, \frac{\rho(t,\vec{x}')}{|\vec{x}-\vec{x}'|} \, . \tag{5.22}$$

So if the charge density $\rho = 0$, for instance for 'pure electromagnetism', in which there is no charged matter ϕ, we have

$$A_0 = 0$$

in Coulomb gauge. On the other hand, if there are charged fields and hence $\rho \neq 0$, then $A_0 \neq 0$.

*** EXERCISE**:

Determine the proportionality factor in (5.22). [**Hint**: use $\nabla^2 \frac{1}{4\pi|\vec{x}|} = \delta^{(3)}(\vec{x})$.]

REMARK: **[Ex 6]**
It is often possible to fix a gauge where[11]

$$A_0 = 0 \, .$$

In this gauge the **energy** of **scalar electrodynamics** (5.4) is

$$E = \int d^3x \left[|\partial_0\phi|^2 + |(\nabla - i\vec{A})\phi|^2 + U(|\phi|^2) + \frac{1}{2g^2}(\vec{E}^2 + \vec{B}^2) \right] \tag{5.23}$$

where

$$E_i = \partial_0 A_i \, , \qquad B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk} \tag{5.24}$$

are the electric and magnetic field.

5.4 $U(1)$ Wilson line and Wilson loop

Let us conclude this chapter with an appetizer of chapters 7 and 8. A good reference for this section is section 15.1 of the book by Peskin and Schroeder [Peskin, 1995].

[11]This is allowed because A_0 has no kinetic term which involves its time derivative in (5.4). A_0 is therefore non-dynamical: it can be determined at all times from A_i and the values of other charged fields in the theory.

We start by recalling that if ϕ is a charged scalar (of charge 1 for definiteness), then its partial derivative is not gauge covariant, that is, it does not transform under a well-defined representation of the $U(1)$ gauge group. You have seen this explicitly in the first term, when you worked out how $\partial_\mu \phi$ transforms under a $U(1)$ gauge transformation (5.6). One can fix this problem by introducing the gauge covariant derivative $D_\mu \phi = (\partial_\mu - iA_\mu)\phi$, which transform covariantly as a field of charge 1 under the gauge transformation (5.6). Hopefully this is all clear by now at a technical level. But why is this, conceptually?

To analyze all the partial derivatives in one fell swoop, let us consider the **total differential** of $\phi(x)$,

$$d\phi(x) = \lim_{\epsilon \to 0} \frac{\phi(x + \epsilon dx) - \phi(x)}{\epsilon} = \partial_\mu \phi(x) dx^\mu \ , \tag{5.25}$$

where I have introduced an infinitesimal book-keeping parameter ϵ in front of the increment dx^μ, so that I could write the total differential as a limit. The final expression, which expresses the total differential of $\phi(x)$ as the 4-vector $\partial_\mu \phi(x)$ contracted with the differential increment dx^μ, follows from Taylor expanding the numerator inside the limit and by taking the limit.

The reason why the total differential (5.25) of ϕ (and hence its partial derivatives) does not transform covariantly under gauge transformations is that the two terms that we are subtracting inside the limit have different gauge transformation properties

$$\begin{aligned} \phi(x + \epsilon dx) &\mapsto e^{i\alpha(x+\epsilon dx)} \phi(x + \epsilon dx) \\ \phi(x) &\mapsto e^{i\alpha(x)} \phi(x) \end{aligned}$$

because $\alpha(x + \epsilon dx) \neq \alpha(x)$.

This problem can be fixed by introducing the 'Wilson line', or, as we will learn in later chapters, the mathematical notion of 'parallel transport'.

Let C be an open curve from point x_1 to point x_2, see figure 5.4. Mathematically, this is a smooth map from an interval to space-time $\mathbb{R}^{1,3}$

$$\begin{aligned} C: \qquad I = [\tau_1, \tau_2] &\mapsto \mathbb{R}^{1,3} \\ \tau \quad &\mapsto x^\mu(\tau) \end{aligned}$$

with $x(\tau_1) = x_1$ and $x(\tau_2) = x_2$ at the endpoints.

The **Wilson line** (of charge 1) along the curve C is defined to be

$$W_C(x_2, x_1) := \exp\left[i \int_{x_1, C}^{x_2} A_\mu(x) dx^\mu \right] \equiv \exp\left[i \int_{\tau_1}^{\tau_2} A_\mu(x(\tau)) \dot{x}^\mu(\tau) d\tau \right] \ , \tag{5.26}$$

where the first integral is the line integral from x_1 to x_2 along curve C, and the second integral is its expression in the parametrization $x^\mu(\tau)$. If C is a closed curve (or a 'loop'), namely if

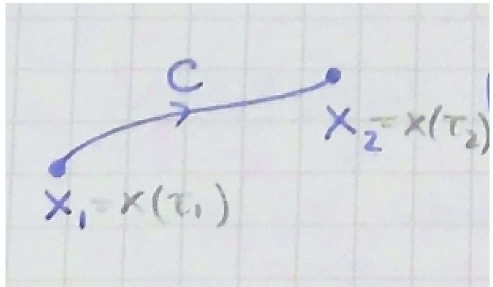

Figure 5.4: An open curve from point x_1 to point x_2.

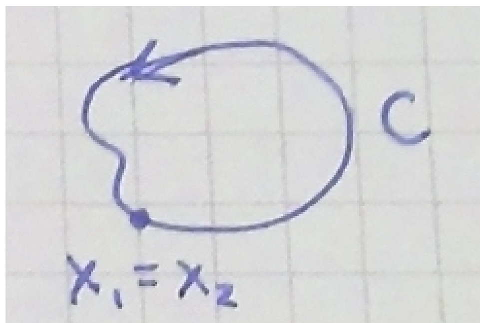

Figure 5.5: A closed curve (or 'loop) with base-point $x_1 = x_2$.

$x_1 = x_2$ as in figure 5.4, then

$$W_C := \exp\left[i \oint_C A_\mu(x) dx^\mu \right] \tag{5.27}$$

is called the **Wilson loop** (of charge 1) along the curve C. By standard results from multivariate calculus, the line integral $\oint_C A_\mu(x)dx^\mu$ only depends on the curve C and not on the base-point $x_1 = x_2$.

Under a $U(1)$ gauge transformation (5.6), we claim that the Wilson line (5.26) transforms as[12]

$$W_C(x_2, x_1) \mapsto e^{i\alpha(x_2)} W_C(x_2, x_1) e^{-i\alpha(x_1)} \ . \tag{5.28}$$

[12]For the gauge group $G = U(1)$, which we are considering here, the Wilson line and the gauge transformations $e^{i\alpha(x_i)}$ commute, so we could have written the gauge transformation of the Wilson line simply as

$$W_C(x_2, x_1) \mapsto e^{i(\alpha(x_2) - \alpha(x_1))} W_C(x_2, x_1) \ .$$

I wrote the result like (5.28) for comparison to the case of a non-abelian gauge group, which we will study later.

Proof.

$$
\begin{aligned}
W_C(x_2,x_1) = e^{i\int_{x_1,\,C}^{x_2} A_\mu dx^\mu} \mapsto\; & e^{i\int_{x_1,\,C}^{x_2}(A_\mu+\partial_\mu\alpha)dx^\mu} \\
=& e^{i\int_{x_1,\,C}^{x_2} A_\mu dx^\mu} e^{i\int_{x_1,\,C}^{x_2} \partial_\mu\alpha dx^\mu} \\
=& W_C(x_2,x_1)e^{i(\alpha(x_2)-\alpha(x_1))} \\
=& e^{i\alpha(x_2)}W_C(x_2,x_1)e^{-i\alpha(x_1)} \;.
\end{aligned}
$$

To go from the second to the third line, we have used the fact that $\partial_\mu\alpha dx^\mu = d\alpha(x)$ is an exact differential, so its integral along a curve C only receives contribution from the boundary (or 'surface') terms.

A corollary of the gauge transformation (5.28) is that the U(1) Wilson loop (5.27) is gauge invariant. To see that, simply set $x_1 = x_2$, or use the fact that the integral of an exact differential along a closed curve vanishes.

Now we can combine the gauge transformation of a charged scalar field and of a Wilson line both of charge 1 to find that the gauge transformation of the product of the charged scalar $\phi(x_1)$ at $x = x_1$ and the Wilson line from x_1 to x_2 along C

$$
\begin{aligned}
W_C(x_2,x_1)\phi(x_1) \mapsto\;\; & e^{i\alpha(x_2)}W_C(x_2,x_1)e^{-i\alpha(x_1)}e^{i\alpha(x_1)}\phi(x_1) \\
=& e^{i\alpha(x_2)}W_C(x_2,x_1)\phi(x_1)
\end{aligned}
\tag{5.29}
$$

is by the same phase $e^{i\alpha(x_2)}$ as for $\phi(x_2)$.

Therefore it makes sense to consider the **total covariant differential**

$$
D\phi(x) = \partial_\mu\phi(x)dx^\mu := \lim_{\epsilon\to 0}\frac{\phi(x+\epsilon dx) - W_{dC}(x+\epsilon dx, x)\phi(x)}{\epsilon}\;, \tag{5.30}
$$

where we have inserted the Wilson line along an infinitesimal line element dC connecting x to $x+\epsilon dx$ in front of $\phi(x)$. This ensures that the terms which are subtracted inside the limit have the same gauge transformation property.

Expanding to first order in ϵ,

$$
\phi(x+\epsilon dx) = \phi(x) + \epsilon\partial_\mu\phi(x)dx^\mu + O(\epsilon^2)
$$

and

$$
\begin{aligned}
W_{dC}(x+\epsilon dx, x) &= \exp\left[i\int_{x,\,dC}^{x+\epsilon dx} A_\mu(x')dx'^\mu\right] \\
&= \exp\left[iA_\mu(x)\epsilon dx^\mu + (\epsilon^2)\right] \\
&= 1 + i\epsilon A_\mu(x)dx^\mu + O(\epsilon^2)\;,
\end{aligned}
\tag{5.31}
$$

and substituting in (5.30) we find

$$\begin{aligned} D\phi(x) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\phi(x)+\epsilon\partial_\mu\phi(x)dx^\mu-\phi(x)-i\epsilon A_\mu(x)\phi(x)dx^\mu+O(\epsilon^2)\right]\,, \\ &= \left(\partial_\mu\phi(x)-i\epsilon A_\mu(x)\phi(x)\right)dx^\mu \equiv D_\mu\phi(x)dx^\mu\,, \end{aligned} \tag{5.32}$$

which precisely reproduces the previous definition (5.5) of the covariant derivative of a scalar field $\phi(x)$ of charge 1!

We will return to this important point in chapters 7 and 8 when we study fibre bundles, associated vector bundles and sections.

<u>**REMARKS**</u>:

1. In QM, the Wilson line $W_C(x_2, x_1)$ is the phase picked up by the wave-function of a charged point particle slowly ('adiabatically') moving from x_1 to x_2 along a curve C in the presence of a gauge field.

2. The Wilson loop (5.27) is gauge invariant and therefore physically observable. It is the phase picked up by the wave-function of a charged point particle slowly moving along a loop C. This phase controls the Aharonov-Bohm effect in QM, a subtle and unexpected form of quantum interference which is due to the fact that the wave-function couples directly to the gauge potential A_μ rather than to the physical electric and magnetic fields $\vec{E}$, $\vec{B}$. As we will see later, if the loop C is not contractible to a point it may happen that $A_\mu \neq 0$ and therefore
$$\oint_C A_\mu dx^\mu \neq 0$$
even if the field strength $F_{\mu\nu} = 0$ vanishes everywhere in the region probed by a quantum-mechanical particle (or by a charged scalar field). This happens for instance in a space of the form $\mathbb{R}^2\backslash p$, for loops which encircle the removed point p.

Time permitting, we might return to the Aharonov-Bohm effect and its underlying geometry in a later chapter. For an accessible summary, see section 10.5.3 of [Nakahara, 2003], up to equation (10.100).

This is enough for our revisitation of $U(1)$ gauge theory. It is now time to follow the steps of Yang and Mills and to study the formulation of gauge theories with non-abelian gauge group.

Chapter 6

Non-abelian gauge theories

In this chapter we will learn how to formulate gauge theories with a non-abelian (that is, non-commutative) gauge group. Non-abelian gauge theories are named **Yang-Mills theories**, after Chen-Ning Yang and Robert Mills, who developed the formalism in 1954 [Yang and Mills, 1954].

The formalism of Yang and Mills became prominent in the late 1960s, and has remain central in modern physics ever since. Non-abelian gauge theories are the **language of the Standard Model** of Particle Physics, and have also established very fruitful **interactions between Physics and Maths**, which have led to numerous developments in both subjects and quite a few Nobel prizes and Fields medals.

We will spend the rest of the term studying the geometry (and some topology) underlying non-abelian gauge field configurations. But let's start by introducing our main characters.

6.1 Compact Lie algebras

This section is mostly a review of material from the previous term, but I will introduce new conventions following the Physics literature. I will also introduce some new terminology and definitions along the way. There will be a number of exercise that I recommend attempting to make sure that you understand the concepts. An excellent reference is section 1.8.1 of Argyres' supersymmetry lecture notes [Argyres, 2001], of which this section is a shameless rip-off.

We start by recalling that a **Lie algebra** $\mathfrak{g}$ is a vector space[1] endowed with an additional

[1] We will only consider finite-dimensional vector spaces. We refer to the dimension of this vector space as the dimension of the Lie algebra, which we denote as $\dim \mathfrak{g}$.

structure, the **Lie bracket**

$$[\ ,\ ]:\ \ \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\ ,$$

which is antisymmetric and bilinear. The vector space has a **basis** $\{t_a\}_{a=1}^{\dim\mathfrak{g}}$ of so called **generators** t_a. In this basis the Lie bracket reads

$$[t_a,t_b]=if_{ab}{}^c t_c \qquad (a,b,c=1,\dots,\dim\mathfrak{g}) \tag{6.1}$$

where $f_{ab}{}^c$ are real **structure constants**, which express the component of the Lie bracket $[t_a,t_b]$ along the generator t_c. Repeated indices are summed over. Note the i in the right-hand side in my conventions. I'll return to why that might be useful shortly. In the rest of the course we will content ourselves with matrix Lie algebras, in which case the Lie bracket is simply the commutator of two matrices, that you are familiar with from Linear Algebra. You can always keep that in mind whenever I use the term Lie bracket. But the abstract definition of Lie algebras and the Lie bracket (6.1) is more general.

The associativity of the Lie bracket is expressed by the **Jacobi identity**

$$[[t_a,t_b],t_c]+[[t_b,t_c],t_a]+[[t_c,t_a],t_b]=0 \tag{6.2}$$

which expressed in the basis of generators is the identity

$$f_{ab}{}^d f_{dc}{}^e+f_{bc}{}^d f_{da}{}^e+f_{ca}{}^d f_{db}{}^e=0 \tag{6.3}$$

for the structure constants.

An r-dimensional **representation** (rep) of $\mathfrak{g}$ is a realization of the generators $\{t_a\}$ as a set of $r\times r$ **matrices** satisfying (6.1), where now [,] is interpreted as the **commutator** of matrices: $[A,B]=AB-BA$. We will often denote an r-dimensional representation as **r** and its generators as $t_a^{(\mathbf{r})}$. If there are multiple representations with the same dimension we will distinguish them by primes or other notation. We might omit the subscript $t_a^{(\mathbf{r})}$ when it is clear from the context which representation we are discussing.

A **compact Lie algebra** is one which can be represented by finite-dimensional **hermitian** matrices:

$$t_a^\dagger=t_a\ . \tag{6.4}$$

<u>REMARK</u>:
The imaginary unit $i=\sqrt{-1}$ in the right-hand side of the Lie bracket (6.1), which is common in the physics literature, is there to ensure the hermiticity of generators of compact Lie algebras (6.4). This convention is convenient to manifest the reality (and positive definiteness) of energy functionals or other physical quantities. The maths literature (and Andreas) typically uses $\tilde{t}_a=it_a$ as generators, which are anti-hermitian for compact Lie algebras: $\tilde{t}_a^\dagger=-\tilde{t}_a$.

It is a theorem that any compact Lie algebra can be decomposed into the direct sum of $u(1)$ Lie alegbras and of simple Lie algebras:

$$\mathfrak{g} = (\oplus_{i=1}^{h} u(1)) \oplus (\oplus_{i=1}^{l} \mathfrak{g}_i) = u(1) \oplus \cdots \oplus u(1) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l \ . \tag{6.5}$$

Let's recall $u(1)$ and semisimple Lie algebras in turn:

1. The $u(1)$ Lie algebra is the compact Lie algebra with a single generator t. By the anti-symmetry of the Lie bracket, we have

   $$[t, t] = 0 \ , \tag{6.6}$$

   so the algebra is **abelian**. Its **irreducible representations** (irreps) are 1-dimensional

   $$t = q\mathbb{1} \qquad (q \in \mathbb{Z}) \tag{6.7}$$

   where the integer q is called the **charge** of the representation and $\mathbb{1}$ is the identity operator.

2. a **simple Lie algebra** is characterised by structure constants such that

   $$f_{ab}{}^{c} \neq 0 \qquad \forall a \tag{6.8}$$

   (or equivalently for all b, or for all c, it turns out).

[Ex 9.1] **<u>* EXERCISE</u>**:

Show that for any representation $\mathbf{r}$ of a simple Lie algebra

$$\mathrm{tr}_{\mathbf{r}}(t_a^{(\mathbf{r})}) = 0 \ , \tag{6.9}$$

where $\mathrm{tr}_{\mathbf{r}}$ is the trace in the representation $\mathbf{r}$, which is nothing but the trace of $r \times r$ matrices.

One can show that there is a basis of generators, which we will adopt from now on, such that

$$\mathrm{tr}_{\mathbf{r}}(t_a^{(\mathbf{r})} t_b^{(\mathbf{r})}) = C(\mathbf{r}) \delta_{ab} \ , \tag{6.10}$$

The real number $C(\mathbf{r})$, which is positive for representations of compact Lie algebras, is called the **quadratic invariant** of the representation $\mathbf{r}$.

<u>REMARKS</u>:

1. There is a **normalization** ambiguity: rescaling

   $$(t_a, f_{ab}{}^{c}, C(\mathbf{r})) \longrightarrow (\lambda t_a, \lambda f_{ab}{}^{c}, \lambda^2 C(\mathbf{r})) \tag{6.11}$$

   by a constant $\lambda \neq 0$ leaves all the previous equations invariant.

2. For the **adjoint representation** $\mathbf{r} = \text{adj}$, which we will review later, equation (6.10) defines the **Killing form**[2]

$$\begin{array}{llll} K\colon & \mathfrak{g}\times\mathfrak{g} & \to & \mathbb{R} \\ & (v,w) & \mapsto & K(v,w) := \text{tr}(\text{ad}_v \circ \text{ad}_w) \end{array} \tag{6.12}$$

where $v = v^a t_a$, $w = w^a t_a$, and ad_x denotes the adjoint action of x on the Lie algebra. The Killing form K is bilinear and symmetric.

3. We can use δ_{ab} / its inverse δ^{cd} to **lower/raise** Lie algebra **indices**.[3]

[Ex 9.2] *** EXERCISE**:

Show that f_{abc} is totally antisymmetric in its indices:

$$f_{abc} = -f_{bac} = -f_{cba}\ . \tag{6.13}$$

We can obtain a **Lie group** G **from a Lie algebra** $\mathfrak{g}$ by applying the **exponential map**[4]

$$\begin{array}{llll} \text{exp:} & \mathfrak{g} & \to & G \\ & \alpha = \alpha^a t_a & \mapsto & g = e^{i\alpha^a t_a} \end{array} \tag{6.14}$$

where $\alpha^a \in \mathbb{R}$.

Substituting the abstract generators t_a of the Lie algebra by their realizations $t_a^{(\mathbf{r})}$ in a representation $\mathbf{r}$, we obtain **representations of the Lie group** G. In an r-dimensional rep $\mathbf{r}$, the group element g is realized as an $r \times r$ **unitary** matrix:

$$g: \quad \phi \mapsto (g\phi) \equiv r(g)\cdot\phi = e^{i\alpha^a t_a^{(\mathbf{r})}}\cdot\phi\ , \tag{6.15}$$

where ϕ and its transormed $(g\phi)$ are r-vectors, while $r(g) = \exp\left[i\alpha^a t_a^{(\mathbf{r})}\right]$ is an $r\times r$ matrix. In components,

$$g: \quad \phi^j \mapsto (g\phi)^j = r(g)^j{}_k\phi^k = (e^{i\alpha^a t_a^{(\mathbf{r})}})^j{}_k\phi^k\ , \tag{6.16}$$

[2] Named after Wilhelm Killing. No humans or animals were harmed in the production of this lecture course.

[3] More precisely, one should use the Killing form and its inverse to lower and raise indices. Since we can pick a basis in which the Killing form is proportional to the identity and since the normalization is arbitrary, we do not lose much by using δ_{ab} instead of the Killing form. It may sound silly to distinguish upper and lower indices if we are raising and lowering them using the identity matrix. I am mentioning this distinction because in a different basis of the Lie algebra the Killing form might be a less trivial non-degenerate symmetric matrix K_{ab}. In such a basis You would use this matrix K_{ab} to lower Lie algebra indices, and its inverse K^{ab} to raise indices. (This is completely analogous to raising/lowering spacetime indices with the Minkowski metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$.)

[4] More precisely, the exponential map of the Lie algebra produces a subgroup whose elements are continuously connected to the identity (*aka* the connected component of the identity).

We will refer to the action (6.15)-(6.16) of a group element on ϕ as the **finite transformation** of ϕ.

Conversely, we recover the action of the Lie algebra if the parameters α^a are infinitesimal. Then to linear order $\phi \mapsto \phi + \delta_\alpha \phi$, with

$$\delta_\alpha \phi^j = i\alpha^a (t_a^{(\mathbf{r})})^j{}_k \phi^k \ , \tag{6.17}$$

which we will refer to as the **infinitesimal transformation** of ϕ.

Next we define the **complex conjugate** representation $\bar{\mathbf{r}}$ of a representation $\mathbf{r}$ as the representation with complex conjugate representation matrix:

$$\bar{r}(g) := r(g)^* \equiv \overline{r(g)} \ , \tag{6.18}$$

where we use star or bar interchangeably to denote complex conjugation.

[Ex 10] *** EXERCISE**:

1. Show explicitly that if the r-vector ϕ transforms in irrep $\mathbf{r}$, then its complex conjugate $\phi^* \equiv \bar{\phi}$, which is also an r-vector, transforms in irrep $\bar{\mathbf{r}}$.

2. Show that, as $r \times r$ matrices,

$$t_a^{(\bar{\mathbf{r}})} = -(t_a^{(\mathbf{r})})^T \ , \tag{6.19}$$

   where the subscript T denotes the transposition of a matrix.

3. Denote $\bar{\phi}_j := (\phi^j)^*$ and construct the **row** r-vector $\phi^\dagger = \bar{\phi}^T = (\bar{\phi}_1, \ldots, \bar{\phi}_r)$. Show that the inner product $\phi^\dagger \phi$ is invariant under the action of G.

Next we introduce the **adjoint representation** adj, which is the $(\dim \mathfrak{g})$ - dimensional irrep defined by

$$(t_a^{(\mathrm{adj})})^b{}_c = i f_{ac}{}^b \qquad (b, c = 1, \ldots, \dim \mathfrak{g}) \ . \tag{6.20}$$

[Ex 11] *** EXERCISE**:

1. Check that (6.18) defines a representation of $\mathfrak{g}$.

2. Recall that the **adjoint action** of the Lie algebra $\mathfrak{g}$ on itself is given by

$$\begin{array}{rccl} \mathrm{ad}: & \mathfrak{g} & \to & \mathfrak{g} \\ & y & \mapsto & \mathrm{ad}_x(y) := [x, y] \end{array} \tag{6.21}$$

for all Lie algebra elements $x \in \mathfrak{g}$. Show that

$$\mathrm{ad}_{t_a}(y^b t_b) = (t_a^{(\mathrm{adj})})^b{}_c y^c t_b \ . \tag{6.22}$$

[**Remark:** this has a nice interpretation: the adjoint representation (6.20) is nothing but the adjoint action of the Lie algebra on itself, expressed in a basis.

3. Show that the quadratic invariant of the adjoint representation is

$$C(\mathrm{adj}) = \frac{f_{abc} f^{abc}}{\dim \mathfrak{g}} \ , \tag{6.23}$$

where Lie algebra indices are raised (/lowered) using δ^{ab} ($/\delta_{ab}$).

Finally we introduce the notion of fundamental representation. Consider a **matrix group** G, that is a group whose elements are square matrices and where the group composition law is matrix multiplication. Let n be the size of the matrices. The **fundamental representation** (or defining representation) of a matrix group G is the representation in which G acts by matrix multiplication:

$$r(g)^i{}_j = g^i_j \tag{6.24}$$

where $g \in G$ is a matrix. We denote the fundamental representation by fund or by $\mathbf{n}$ according to its dimension.

[Ex 12] * **EXERCISE**:

Let $\mathrm{Mat_n(F)}$ denote $n \times n$ matrices whose entries are in the field F, and $\mathbb{1}_n$ the $n \times n$ identity matrix. The **classical compact simple Lie groups** are[5]

$$\begin{aligned} SU(N) \quad &= \{g \in \mathrm{Mat_N(\mathbb{C})} \mid \mathrm{g^\dagger g} = \mathbb{1}_\mathrm{N} \ , \ \ \det \mathrm{g} = 1\} \\ SO(N) \quad &= \{g \in \mathrm{Mat_N(\mathbb{R})} \mid \mathrm{g^T g} = \mathbb{1}_\mathrm{N} \ , \ \ \det \mathrm{g} = 1\} \\ USp(2N) \quad &= \{g \in \mathrm{Mat_{2N}(\mathbb{C})} \mid \mathrm{g^\dagger g} = \mathbb{1}_\mathrm{2N} \ , \ \ \mathrm{g^T J g} = \mathbb{1}_\mathrm{2N}\} \end{aligned}$$

where the $(2N) \times (2N)$ antisymmetric matrix

$$J = \begin{pmatrix} 0_N & \mathbb{1}_N \\ -\mathbb{1}_N & 0_N \end{pmatrix}$$

is called the symplectic form.

[5] The term 'classical' comes from Cartan's classification of simple Lie algebras. The classification consists of a few classical families each labelled by an integer N, which are the Lie algebras of the above Lie groups, along with a few exceptional Lie algebras which are not of matrix type. We will ignore the latter in this course

1. Characterize the Lie algebras $su(N)$, $so(N)$, and $usp(2N)$ as vector spaces of matrices subject to certain linear conditions, which you should find.
   [**Hint**: You can assume that a group element takes the exponential form $g = \exp(i\alpha^a t_a)$ and Taylor expand for infinitesimal α.]

2. Find the generators of the fundamental representation fund and its complex conjugate rep $\overline{\text{fund}}$ (the so called antifundamental representation) for $G = SU(N)$, $SO(N)$, $USp(2N)$. For $G = SO(N)$, $USp(2N)$, show that fund and $\overline{\text{fund}}$ are representations, namely

   $$t_a^{(\overline{\text{fund}})} = V t_a^{(\text{fund})} V^{-1} \qquad \forall a$$

   for some invertible matrix V.[6]

[6]For compact Lie groups and algebras, V is unitary: $V^\dagger = V^{-1}$.

Chapter 7

Fibre bundles, connections, and curvature

Chapter 8

Coupling to charged fields: vector bundles and sections

Chapter 9

Topological solitons and instantons

Chapter 10

Spinors and index theorems

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