PROBLEMS CLASS 5

$\boxed{\text{Ex 4}}$ Let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

1. Show that the "vacuum Maxwell eqns" (i.e. with $J^\mu = 0$) have plane wave solutions of the form

wavenumber 4-vector
(momentum 4-vector)

$$A_\mu = Re\left(p_\mu e^{ik_\nu x^\nu}\right)$$

polarization vector

for constant vectors p_μ, k_μ. Which conditions must p_μ, k_μ obey?

The EoM is

$$0 = \partial^\nu F_{\mu\nu} = \partial^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0 .$$

NOTATION:
$a \cdot b \equiv a^\mu b_\mu$
$a^2 = a \cdot a$

Letting $A_\mu = Re\left(p_\mu e^{ik\cdot x}\right)$ gives

$$0 = Re\left(\underbrace{\left((ik^\nu)(ik_\mu)p_\nu - (ik^\nu)(ik_\nu)p_\mu\right)}\, e^{ik\cdot x}\right) \qquad \forall x$$

so the plane wave is a solution provided that

$$-k\cdot p\, k_\mu + k^2 p_\mu = 0 \qquad \forall \mu = 0,1,2,3 .$$

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2. Taking into account the freedom of gauge transformations, how many independent components of p_μ appear in a general plane wave solution? [Hint: try $\alpha = Re(c\, e^{ik\cdot x})$] ($c$ const.)

Under a gauge transfo,

~~Note: this gauge transfo keeps the plane wave a plane wave~~

$$A_\mu \longmapsto A_\mu + \partial_\mu \alpha$$

$$\Rightarrow \mathrm{Re}\left(p_\mu e^{ik\cdot x}\right) \longmapsto \mathrm{Re}\left((p_\mu + ick_\mu)\, e^{ik\cdot x}\right)$$

i.e.
$$p_\mu \longmapsto p_\mu + ick_\mu \,. \qquad (*)$$

Taking $c \in i\cdot\mathbb{R}$ (purely imaginary), this allows us to eliminate the component of p_μ parallel to k_μ ("longitudinal component"). Then

$$k\cdot p\, k_\mu = k^2 p_\mu$$

implies that
$$k\cdot p = k^2 = 0 \,. \qquad (**)$$

- $k^2 \equiv k^\mu k_\mu = 0$ is the dispersion relation of electromagnetism, which tells us that $\underset{\omega}{\overset{}{\underset{|||}{k_0}}} = \pm |\vec{k}|$.
- $k\cdot p \equiv k^\mu p_\mu = 0$ says that the polarization vector p_μ is orthogonal (with the Minkowski metric) to the wave 4-vector k_μ (or equivalently, momentum) of the electromagnetic wave.

- The gauge freedom $(*)$ eliminates the longitudinal component of p_μ.
- The equation $p\cdot k = 0$ eliminates another component.

$$\Rightarrow \left[\#\text{ of independent components ("degrees of freedom") of } p_\mu\right] = 4 - 1 - 1 = 2 .$$

<u>NOTE</u>: using $(**)$ in $(*)$ we find

$$0 = k\cdot p \longmapsto k\cdot p + ick^2 = k\cdot p = 0$$
$$p^2 \longmapsto (p+ick)^2 = p^2 + 2ick\cdot p - c^2k^2 = p^2$$

so $k\cdot p$ and p^2 are gauge invariant.

$\boxed{\text{Ex 6}}$

Consider a complex scalar field ϕ with Lagrangian density

$$\mathcal{L} = \underbrace{-|\partial_\mu \phi|^2}_{\substack{\equiv \\ -\partial^\mu \bar{\phi}\, \partial_\mu \phi \,=\, |\partial_0 \phi|^2 - |\vec{\nabla}\phi|^2}} - V(\phi, \bar{\phi}) \equiv -|\partial_\mu \phi|^2 - \lambda\left(|\phi|^2 - a^2\right)^2 \qquad (\lambda, a > 0)$$

1. Show that the energy (or 'Hamiltonian') is

$$E = \int d^3x \left(|\partial_0 \phi|^2 + \underbrace{|\partial_i \phi|^2}_{\equiv} + V(\phi, \bar{\phi}) \right)$$

NOTATIONS: $\partial_i \bar{\phi}\, \partial_i \phi = \sum_{i=1}^{3} |\partial_i \phi|^2 \equiv |\vec{\nabla}\phi|^2$ (All these notations are fine)

The Hamiltonian is the integral over space $\mathbb{R}^3$ of the Hamiltonian density

$$\mathcal{H} = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}}_{=\,\pi_\phi} \partial_0 \phi + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_0 \bar{\phi})}}_{=\,\pi_{\bar{\phi}}} \partial_0 \bar{\phi} - \mathcal{L}$$

(conjugate momenta to $\phi, \bar{\phi}$)

$$= (\overline{\partial_0 \phi})(\partial_0 \phi) + (\partial_0 \phi)(\overline{\partial_0 \phi}) - (\overline{\partial_0 \phi})(\partial_0 \phi) + \vec{\nabla}\bar{\phi} \cdot \vec{\nabla}\phi + V(\bar{\phi}, \phi)$$

$$= |\partial_0 \phi|^2 + |\vec{\nabla}\phi|^2 + V(\bar{\phi}, \phi)$$

$$\Rightarrow E = \int d^3x \left(|\partial_0 \phi|^2 + |\vec{\nabla}\phi|^2 + V(\bar{\phi}, \phi) \right)$$

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2. Find the configurations of least energy ("vacua"/"ground states") and show that they parametrize a circle in field space.

All 3 terms in $\mathcal{H}$ are separately ≥ 0, so the energy is minimized by setting them to zero separately:

$$\partial_0 \phi = 0, \quad \vec{\nabla}\phi = 0, \quad |\phi|^2 = a^2 .$$

Hence the ground states/vacua are constant field configurations that minimize the scalar potential. Here

$$\{ \underbrace{\phi = a e^{i\beta}}_{\text{constants}} \mid \beta \sim \beta + 2\pi \} \cong S^1$$

(Circle of radius a in the complex ϕ plane)

3. Show that any two vacua are related by a global U(1) transformation.

- Global (internal) U(1) transfo:

(*) $$\phi \mapsto e^{i\alpha}\phi = \phi' \qquad (\alpha \sim \alpha + 2\pi)$$

- Vacuum 1: $\phi = a e^{i\beta_1} \equiv \phi_1$

  Vacuum 2: $\phi = a e^{i\beta_2} \equiv \phi_2$

Perform (*) with $\alpha = \beta_2 - \beta_1$:

$$\phi_1 = a e^{i\beta_1} \longmapsto \phi_1' = e^{i(\beta_2 - \beta_1)} a e^{i\beta_1} = a e^{i\beta_2} = \phi_2 .$$

Ex 7 Show that the commutator of two covariant derivatives $D_\mu = \partial_\mu - iA_\mu$ is given by

$$-i F_{\mu\nu} \equiv [D_\mu, D_\nu] = -i(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

We need to use

$$[f,g] = 0 \qquad \forall\ f,g \text{ functions of } x$$

$$[\partial_\mu, f] = \partial_\mu f - f\partial_\mu = (\partial_\mu f)\mathbb{1} + f\partial_\mu - f\partial_\mu = (\partial_\mu f) \qquad \forall \text{ functions } f \text{ of } x$$

$$[\partial_\mu, \partial_\nu] = 0$$

where the above are viewed as differential operators (functions of x act by multiplication), and

$$[X, Y] = Z \qquad \text{as operators}$$

iff
$$[X, Y]\psi = Z\psi \qquad \forall \text{ smooth functions } \psi \text{ of } x.$$

Indeed

$$[f, g]\psi = fg\psi - gf\psi = 0$$

$$[\partial_\mu, f]\psi = \partial_\mu(f\psi) - f(\partial_\mu\psi) = (\partial_\mu f)\psi + \cancel{f(\partial_\mu\psi)} - \cancel{f(\partial_\mu\psi)} = (\partial_\mu f)\psi$$

$$[\partial_\mu, \partial_\nu]\psi = \partial_\mu\partial_\nu\psi - \partial_\nu\partial_\mu\psi = 0.$$

Then

$$F_{\mu\nu}\mathbb{1} = i[D_\mu, D_\nu] = i[\partial_\mu - iA_\mu, \partial_\nu - iA_\nu]$$

$$= i\Big(\underset{=0}{[\partial_\mu, \partial_\nu]} - i[\partial_\mu, A_\nu] - i[A_\mu, \partial_\nu] - \underset{=0}{[A_\mu, A_\nu]}\Big)$$

$$= [\partial_\mu, A_\nu] - [\partial_\nu, A_\mu] = (\partial_\mu A_\nu - \partial_\nu A_\mu)\mathbb{1}.$$