GMP PROBLEMS CLASS 7 2/3/2023

Ex 24

1. Consider a rep $\underline{r}$ of a compact Lie group G, w/ generators $t_a^{(\underline{r})}$. Show that the set of generators

$$t_a^{(\bar{\underline{r}})} := -\left(t_a^{(\underline{r})}\right)^T$$

defines another rep of G, which is called the complex conjugate rep $\bar{\underline{r}}$.

We need to show that (drop underlines for ease of notation)

$$[t_a^{(r)}, t_b^{(r)}] = i f_{ab}{}^c t_c^{(r)} \Rightarrow [t_a^{(\bar{r})}, t_b^{(\bar{r})}] = i f_{ab}{}^c t_c^{(\bar{r})}.$$

$$[t_a^{(\bar{r})}, t_b^{(\bar{r})}] = [-(t_a^{(r)})^T, -(t_b^{(r)})^T] = [t_b^{(r)}, t_a^{(r)}]^T$$

$$= -[t_a^{(r)}, t_b^{(r)}]^T = -i f_{ab}{}^c (t_c^{(r)})^T = i f_{ab}{}^c t_c^{(\bar{r})} .$$

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2. Show that if the column vector ϕ transforms in the irrep $\underline{r}$, then its complex conjugate $\phi^* = \bar{\phi}$, which is also a column vector, transforms in the complex conjugate irrep $\bar{\underline{r}}$.

$$\phi \mapsto \underbrace{e^{i\alpha^a t_a^{(r)}}}_{r(g),\ g = e^{i\alpha^a t_a}} \phi$$

$$\Rightarrow \bar{\phi} \mapsto e^{-i\alpha^a \overline{t_a^{(r)}}} \bar{\phi} = e^{i\alpha^a (-t_a^{(r)})^T} \bar{\phi} = e^{i\alpha^a t_a^{(\bar{r})}} \bar{\phi}.$$

$\alpha^a \in \mathbb{R}$ $\qquad$ $\underbrace{t_a^{(r)\dagger}}_{(\overline{t_a^{(r)}})^T} = t_a^{(r)}$ for reps of compact Lie groups

3. Denote $\overline{\phi}_j := (\phi^j)^*$, $\phi^\dagger = \overline{\phi}^T = (\overline{\phi}_1, \dots, \overline{\phi}_r)$ ← row vector

Show that the inner product $\phi^\dagger \phi$ is invariant under the action of G.

$$\phi^\dagger \mapsto \left(e^{i\alpha^a t_a^{(r)}} \phi\right)^\dagger = \phi^\dagger e^{-i\alpha^a t_a^{(r)}} \quad t_a^{(r)} = (t_a^{(r)})^\dagger$$

$$\Rightarrow \phi^\dagger \phi \mapsto \phi^\dagger \underbrace{e^{-i\alpha^a t_a^{(r)}} e^{i\alpha^b t_b^{(r)}}}_{= e^{-i\alpha^{(r)}} e^{i\alpha^{(r)}} = \mathbb{1}_{r\times r}} \phi = \phi^\dagger \phi .$$

matrices

$\boxed{\text{Ex } 25}$

1. The adjoint action of the Lie algebra g on itself is

$$\begin{aligned} ad_x : g &\longrightarrow g \\ y &\longmapsto ad_x(y) := [x, y] \end{aligned}$$

$\forall x \in g$. Show that

$$ad_{t_a}(y^b t_b) = (t_a^{(adj)})^b{}_c \, y^c t_b , \quad \text{where} \quad (t_a^{(adj)})^b{}_c = i f_{ac}{}^b$$

are the generators of the adjoint rep.

$$ad_{t_a}(y^b t_b) = [t_a, y^b t_b] = y^b [t_a, t_b] = y^b \cdot i f_{ab}{}^c t_c$$

$$= i f_{ac}{}^b y^c t_b = (t_a^{(adj)})^b{}_c \, y^c t_b .$$

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2. Express the Killing form

$$K_{ab} = tr_{adj}\left(t_a^{(adj)} t_b^{(adj)}\right)$$

in terms of the structure constants.

$$K_{ab} = (t_a^{(adj)})^d{}_c \, (t_b^{(adj)})^c{}_d = i f_{ac}{}^d \cdot i f_{bd}{}^c$$

$$= - f_{ac}{}^d f_{bd}{}^c = f_{ac}{}^d f_{db}{}^c \, .$$

3. Show that, in a basis where the Killing form is

$$K_{ab} = C(\text{adj}) \delta_{ab} \, ,$$

the quadratic invariant $C(\text{adj})$ of the adjoint rep is given by

$$C(\text{adj}) = \frac{\delta^{ab} f_{acd} f_b{}^{cd}}{\dim \mathfrak{g}} \, .$$

$$C(\text{adj}) \delta_{ab} = f_{ac}{}^d f_{db}{}^c$$

$\delta^{ba} \times \uparrow$: $$C(\text{adj}) \underbrace{\delta^a_a}_{= \dim \mathfrak{g}} = \delta^{ba} f_{ac}{}^d f_{db}{}^c \underset{\uparrow \; f_{db}{}^c = -f_{bd}{}^c}{=} - \delta^{ab} f_{ac}{}^d f_{bd}{}^c$$

Lower & Raise d (either with K or δ, both symmetric)

$$= - \delta^{ab} f_{acd} f_b{}^{dc} \underset{\uparrow \; f_{acd} = -f_{adc}}{=} \delta^{ab} f_{adc} f_b{}^{dc}$$

$$\underset{\uparrow \; \text{relabel } c \leftrightarrow d}{=} \delta^{ab} f_{acd} f_b{}^{cd}$$

$$\Rightarrow C(\text{adj}) = \frac{\delta^{ab} f_{acd} f_b{}^{cd}}{\dim \mathfrak{g}} \, .$$

Ex 28

By considering infinitesimal gauge transformations ($|\alpha^a| \ll 1$)

$$g = e^{i\alpha^a t_a} \equiv e^{i\alpha} = \mathbb{1} + i\alpha + O(\alpha^2)$$

and Taylor expanding finite gauge transformations to leading order in $\alpha \in \mathfrak{g} = \mathrm{Lie}(G)$, show that the infinitesimal gauge variations of the fields are

$$\delta_\alpha \phi = i\alpha\phi$$
$$\delta_\alpha A_\mu = i[\alpha, A_\mu] + \partial_\mu \alpha$$
$$\delta_\alpha F_{\mu\nu} = i[\alpha, F_{\mu\nu}]$$

where $\phi \mapsto \phi + \delta_\alpha \phi + O(\alpha^2)$ and so on.

Finite gauge transformations:

$$\phi \mapsto \phi' = g \cdot \phi = e^{i\alpha}\phi$$
$$A_\mu \mapsto A'_\mu = g A_\mu g^{-1} + i g(\partial_\mu g^{-1})$$
$$= e^{i\alpha} A_\mu e^{-i\alpha} + i e^{i\alpha}(\partial_\mu e^{-i\alpha})$$

$$F_{\mu\nu} \mapsto F'_{\mu\nu} = g F_{\mu\nu} g^{-1} = e^{i\alpha} F_{\mu\nu} e^{-i\alpha}$$

Taylor expand to first order:

$$\phi \mapsto (\mathbb{1} + i\alpha + O(\alpha^2))\phi = \phi + i\alpha\phi + O(\alpha^2)$$

$$\Rightarrow \delta_\alpha \phi = i\alpha\phi$$

$$A_\mu \mapsto (\mathbb{1} + i\alpha + O(\alpha^2)) A_\mu (\mathbb{1} - i\alpha + O(\alpha^2)) + i(\mathbb{1} + i\alpha + O(\alpha^2))(\partial_\mu(\mathbb{1} - i\alpha + O(\alpha^2)$$
$$= A_\mu + i\alpha A_\mu - iA_\mu\alpha + i(-i)(\partial_\mu\alpha) + O(\alpha^2)$$
$$= A_\mu + i[\alpha, A_\mu] + (\partial_\mu\alpha) + O(\alpha^2)$$

$$\Rightarrow \delta_\alpha A_\mu = i[\alpha, A_\mu] + \partial_\mu\alpha.$$

(Similarly, $\delta_\alpha F_{\mu\nu} = i[\alpha, F_{\mu\nu}]$)